Essential norm of Cesàro operators on L^p and Cesàro spaces

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Abstract

In this paper, we consider the Cesàro-mean operator Γ acting on some Banach spaces of measurable functions on $(0,1)$, as well as its discrete version on some sequences spaces. We compute the essential norm of this operator on $L^p([0,1])$, for $p \in (1,+\infty]$ and show that its value is the same as its norm: $p/(p-1)$. The result also holds in the discrete case. On Cesàro spaces the essential norm of Γ turns out to be 1. At last, we introduce the Müntz-Cesàro spaces and study some of their geometrical properties. In this framework, we also compute the value of the essential norm of the Cesàro operator and the multiplication operator restricted to those Müntz-Cesàro spaces.

Key words: Cesàro spaces, Cesàro operator, Müntz spaces, compact operator, essential norm, Multiplication operator.
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1 Introduction

Throughout this paper, we denote by $\mathcal{C} = \mathcal{C}([0,1])$ the space of continuous functions on $[0,1]$ equipped with the supremum norm and by $\mathcal{C}_0$ the subspace of $\mathcal{C}$ (resp. c) of functions vanishing at zero (resp. the space of convergent sequences). For $p \in [1,+\infty)$ and (Ω,μ) a measure space, we denote as usual by $L^p(\mu) = L^p(\Omega,\mu)$ the Banach space of μ-measurable functions f on Ω such that $\|f\|_p = (\int_\Omega |f|^p d\mu)^{1/p} < \infty$. In particular when μ is the Lebesgue measure on

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$[0,1]$ (resp. ν the counting measure on $\mathbb{N}$), we just use L^p (resp. ℓ^p). For $p=\infty$, we denote by L^∞ (resp. ℓ^∞) the space of essentially bounded measurable functions on $[0,1]$ (resp. the space of bounded sequences) endowed with its usual norm.

One of the interesting questions in operator theory is to check whether an operator acting on a Banach space is compact or not. This can be essentially done by computing the essential norm, denoted by $\|\cdot\|_e$ (resp. $\|\cdot\|_{e,w}$), of the underlying operator to see how far is its distance from the set of compact operators (resp. weakly compact operators). This question has been adressed by many authors and has been investigated for different operators acting on several Banach spaces.

In this paper, we are interested in the Cesàro operator defined on Lebesgue spaces and on Cesàro spaces. Recall that these latter, denoted by Ces_p ($p \in [1,\infty]$), are defined on functions as the set of Lebesgue measurable complex functions f on $I=[0,1]$ endowed with the norm

$$\|f\|_{C(p)} := \left[\int_I \left(\frac{1}{x}\int_0^x |f(t)|dt\right)^p dx\right]^{1/p} < \infty \quad \text{for } 1 \leq p < \infty$$

and

$$\|f\|_{C(\infty)} := \sup_{x\in I, x>0} \left(\frac{1}{x}\int_0^x |f(t)|dt\right) < \infty \quad \text{for } p=\infty.$$

Also, they are denoted by ces_p on complex sequences and defined as the set of all $u=(u_k)_{k\geq 1}$ such that

$$\|u\|_{c(p)} := \left[\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|u_k|\right)^p\right]^{1/p} < \infty \quad \text{when } 1 \leq p < \infty$$

and

$$\|u\|_{c(\infty)} := \sup_{n\geq 1} \frac{1}{n}\sum_{k=1}^{n}|u_k| < \infty \quad \text{when } p=\infty.$$

Note that Ces_1 is an $L^1(w)$-space with the weight $w(t)=\log(\frac{1}{t})$, and that $\mathrm{ces}_1=\{0\}$ (see [AM, Theorem 1]). The Cesàro operator is the map $\Gamma : L^1_{loc}([0,1)) \to \mathcal{C}((0,1))$ defined for any function f by

$$\Gamma(f)(x) = \frac{1}{x}\int_0^x f(t)dt \quad \text{where } x\in(0,1).$$

Clearly, this operator maps the space L^p ($p>1$) to itself through the Hardy inequality but it does not preserve the space L^1. In this case, we use the notation Γ_p (resp. $\Gamma_{C(p)}$) as the restriction of Γ to L^p (resp. Ces_p). Similarly, the definition of Γ can be transposed to the Cesàro sequence operator $\gamma_p : \ell^p \to \ell^p$ ($p\in(1,+\infty]$) by

$$\gamma_p((u_k)_k) = \left(\frac{1}{n}\sum_{k=1}^{n} u_k\right)_n .$$

In the same way, we denote by $\gamma_{c(p)}$ the extension of γ_p to the set ces_p.

The Cesàro operators and Cesàro spaces were already studied on many aspects, but lately the topic has received a particular interest, see for instance [ABR], [CR], [AHLM] and the survey [AM]. Let us quote the result in [CR] where it is proved that the non compactness property of the Cesàro operator acting on a Cesàro space is a general phenomenon. However, the continuity does always hold.

In this paper, we are interested in studying the default of compactness (resp. of weak compactness) of those operators. Concretely, we compute their essential norms (resp. generalized essential norm) as well as their n^{th}-approximation numbers. Recall that this number, denoted by $a_n(T)$, is the distance from any operator T to the set of all bounded linear operators of rank at most $n-1$. In other words, it measures the way of an operator to be of finite rank.

In the same context, we consider the restriction of the Cesàro operators to particular spaces, namely the Müntz spaces [M] in order to check whether the above properties still hold. It turns out that the geometry of such spaces plays a fundamental role to get the compacity in some cases. Here, we recall that a Müntz space M_Λ^E is the closure in some Banach space E of the linear space spanned by the monomials x^{λ_n} where $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ is an increasing sequence of positive numbers satisfying the Müntz condition $\sum_{n\geq 1} 1/\lambda_n < \infty$.

The paper is divided into four parts. In Section 2, we show that the classical Müntz theorem holds for the Cesàro spaces. Namely, the space $M_\Lambda^{\mathrm{Ces}_p}$ is a strict subspace of Ces_p if and only if the Müntz condition holds (see Theorem 2.3). Moreover, we state a theorem *à la Clarkson-Erdös* for those Müntz-Cesàro spaces (see Propostion 2.7), as well as a bounded Bernstein-type inequality (see Proposition 2.8). This yields to the following property of the Müntz-Cesàro spaces: For any bounded sequence belonging to those spaces, there exist a function f and a subsequence that converges uniformly to f on every compact set of $[0,1)$ (see Corollary 2.9). In Section 3, we state some general criteria to get a lower estimate for the essential norm of a bounded operator $T : X \to Y$ acting between arbitrary Banach spaces X and Y. We then specify our result when Y is an $L^p(\mu)$ space or a $\mathcal{C}(K)$ space. We use these tools in the sequel. In Section 4 and based on the previous section, we first compute the essential norm of the continuous Cesàro operator (resp. discrete) acting on the Lebesgue space L^p (resp. ℓ^p) for $p \in [1,+\infty]$. We find that it is equal to $p' = \frac{p}{p-1}$ for $p \in]1,+\infty[$ (see Theorems 4.1 and 4.2) while it is 1 when $p = \infty$ (Theorems 4.3 and 4.4). We also deduce the approximation numbers of these operators. In the second part of Section 4, we study the Cesàro operators defined on the Cesàro spaces Ces_p and ces_p and show in Theorem 4.7 that their essential norms are all equal to 1. We also consider the restriction of those operators to the Müntz-Cesàro spaces $M_\Lambda^{\mathrm{Ces}_p}$ for $p \in [1,+\infty]$ and prove in Theorem 4.8 that the essential norm is 1 for $p \in [1,+\infty)$ and to $\frac{1}{2}$ for $p = \infty$. The last section is devoted to the study of the compactness of the multiplication operator on the Cesàro function spaces, $T_\psi : \mathrm{Ces}_p \to \mathrm{Ces}_p$ defined by $T_\psi(f) = f\psi$, for $p \in [1,\infty]$ and $\psi \in L^\infty$. We prove that $\|T_\psi\|_e = \|\psi\|_\infty$ and when one restricts to the Müntz-Cesàro subspaces, $T_{\psi,\Lambda} : M_\Lambda^{\mathrm{Ces}_p} \to \mathrm{Ces}_p$ that $\|T_{\psi,\Lambda}\|_e = |\psi(1)|$ if ψ is continuous at 1 (see Theorems 5.2 and 5.4).

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2 Müntz theorem in Cesàro spaces

In this section, we show that the Müntz theorem holds in Cesàro spaces Ces_p by using the Müntz theorems in $\mathcal{C}$ and in L^1 (see [M], [BE]). Hence, we can define the Müntz-Cesàro spaces and study some of their properties. We start with the following lemma which shows that the Cesàro function spaces are embedded into $L^1([0,a])$, for $a \in (0,1)$. We will use this lemma to prove the density of the continuous functions in the Cesàro function spaces.

Lemma 2.1. *Let* $p \in [1,+\infty]$, *and* $0 < a < b \leq 1$. *Then, the Cesàro function spaces satisfy the following bounded inclusions*

$$\mathcal{C} \subset \mathrm{Ces}_p \subset L^1([0,a]).$$

More precisely, for all $f \in \mathrm{Ces}_p$, *we have*

$$\int_0^a |f(t)|dt \leq \frac{b}{(b-a)^{1/p}}\Big(\int_0^b \big(\Gamma(|f|)(x)\big)^p dx\Big)^{1/p} \leq \frac{b}{(b-a)^{1/p}}\|f\|_{C(p)}. \quad (1)$$

Moreover, if $p=1$, *we have*

$$\|f\|_{L^1([0,a])} \leq \frac{1}{\ln\left(\frac{b}{a}\right)}\int_0^b \Gamma(|f|)(x)dx \leq \frac{1}{\ln\left(\frac{b}{a}\right)}\|f\|_{C(1)}. \quad (2)$$

We point out that when a is close to 1 (more precisely when a is larger than some a_p depending on p only), the sharpest version of the previous inequalities is obtained for $b=1$.

Proof. For a continuous function f on $[0,1]$, we obviously have $\|f\|_{C(p)} \leq \|f\|_\infty$. For the right side inclusion, we let $p \in [1,+\infty)$ and $f \in \mathrm{Ces}_p$ to estimate the norm:

$$\begin{aligned}\|f\|^p_{C(p)} &\geq \int_0^b \Big(\frac{1}{x}\int_0^x |f(t)|dx\Big)^p dx\\ &\geq \frac{1}{b^p}\int_a^b \Big(\int_0^x |f(t)|dt\Big)^p dx\\ &\geq \frac{(b-a)}{b^p}\|f\|^p_{L^1([0,a])}.\end{aligned}$$

In the case $p=+\infty$, we obtain easily $\|f\|_{L^1([0,a])} \leq a\|f\|_{C(\infty)}$ and this holds also for $a = 1$. For $p = 1$ and $f \in \mathrm{Ces}_1$, we use Fubini's theorem to obtain

$$\begin{aligned}\|f\|_{C(1)} &\geq \int_0^b \Big(\frac{1}{x}\int_0^x |f(t)|dt\Big)dx\\ &= \int_0^b |f(t)|dt \int_t^b \frac{dx}{x}\\ &= \int_0^b \ln\left(\frac{b}{t}\right)|f(t)|dt\\ &\geq \ln\left(\frac{b}{a}\right)\|f\|_{L^1([0,a])}.\end{aligned}$$

□

The following proposition gives an interesting property of the Cesàro spaces. We will need this property to state the Müntz theorem in these spaces.

Proposition 2.2. *For $p \in [1, +\infty)$, the space of continuous functions, as well as $\mathcal{C}_0$, is dense in Ces_p. The statement is false for $p = +\infty$ since the space Ces_∞ is not separable.*

Proof. For $p = 1$, the first assertions are clear as Ces_1 is a weighted L^1-space. Now, we focus on the case where $p \in (1, \infty)$. For this, we fix $\varepsilon > 0$ and a function $f \in \mathrm{Ces}_p$. As $\Gamma(|f|) \in L^p$, then there exists a number $\delta \in (0, \frac{1}{3})$ satisfying

$$\int_0^{2\delta} (\Gamma(|f|)(x))^p dx \leq \varepsilon^p \quad \text{and} \quad \int_{1-\delta}^1 (\Gamma(|f|)(x))^p dx \leq \varepsilon^p \ .$$

By applying Inequality (1) of Lemma 2.1 with $a = \delta$ et $b = 2\delta$, we obtain

$$\|f\|_{L^1([0,\delta])} \leq \frac{2\delta}{\delta^{1/p}}\Big(\int_0^{2\delta} \big(\Gamma(|f|)(x)\big)^p dx\Big)^{1/p} \leq 2\delta^{1-\frac{1}{p}}\varepsilon \ .$$

Since the space of continuous functions on $[\delta, 1-\delta]$ and vanishing at points δ and $1-\delta$ is dense in $L^1([\delta, 1-\delta])$, there exists a continuous function φ on $[0, 1]$ such that

(i) $\varphi(t) = 0$ for any $t \in [0, \delta] \cup [1-\delta, 1]$;

(ii) $\|f - \varphi\|_{L^1([\delta,1-\delta])} < \delta^{1-\frac{1}{p}}\varepsilon$.

Then we get $\|f - \varphi\|_{L^1([0,1-\delta])} \leq 3\delta^{1-\frac{1}{p}}\varepsilon =: \varepsilon'$. This gives for any $x \in (0, 1-\delta]$ the following

$$\Gamma(|f-\varphi|)(x) = \frac{1}{x}\int_0^x |f-\varphi|(t)dt \leq \frac{\varepsilon'}{x} = \frac{3\delta^{1-\frac{1}{p}}\varepsilon}{x}.$$

Hence, we compute

$$\begin{aligned}\|f-\varphi\|^p_{C(p)} = & \int_0^{\delta} (\Gamma(|f|)(x))^p dx + \int_{\delta}^{1-\delta} (\Gamma(|f-\varphi|)(x))^p dx + \\ & \int_{1-\delta}^{1} (\Gamma(|f-\varphi|)(x))^p dx \\ & \leq \varepsilon^p + \varepsilon'^p \int_{\delta}^{1-\delta} \frac{dx}{x^p} + \int_{1-\delta}^{1} \frac{1}{x^p} \Big(\int_0^{1-\delta} |f-\varphi|(t)dt + \int_{1-\delta}^{x} |f(t)|dt \Big)^p dx \\ & \leq \varepsilon^p + \frac{\varepsilon'^p}{(p-1)\delta^{p-1}} + \int_{1-\delta}^{1} \frac{1}{x^p} \Big(\varepsilon' + \int_0^x |f(t)|dt \Big)^p dx \\ & \leq \varepsilon^p + \frac{3^p}{p-1}\varepsilon^p + 2^p \Big(\varepsilon'^p \int_{1-\delta}^{1} \frac{dx}{x^p} + \int_{1-\delta}^{1} \Gamma(|f|)(x)^p dx \Big) \\ & \leq \varepsilon^p + \frac{3^p}{p-1}\varepsilon^p + 2^p \Big(\Big(\frac{9\delta\varepsilon}{2} \Big)^p + \varepsilon^p \Big) \\ & \leq \Big(1 + \frac{3^p}{p-1} + \frac{3^p}{2} + 2^p \Big) \varepsilon^p. \end{aligned}$$

Therefore, we deduce that the space $\mathcal{C}_0$ is dense in Ces_p as φ vanishes at the point 0. We point out that in the case $p = 1$, the same proof works by using Inequality (2) with $a = \delta$ et $b = \sqrt{\delta}$. Finally, to check the non-separability of Ces_∞, we just mention a short argument to justify it: For the sequence of disjoint intervals $(I_n)_n$ given by $I_n = \Big(\frac{1}{(n+1)!}, \frac{1}{n!} \Big]$, the operator $\Phi : \ell^\infty \to \mathrm{Ces}_\infty$ defined by

$$\Phi\big((a_n)_{n\geq 2}\big) = \sum_n a_n \mathbb{1}_{I_n}$$

embedds isomorphically ℓ^∞ into a subspace of Ces_∞. Indeed, for any $a = (a_n)_{n\geq 2}$ we have $\|\Phi(a)\|_{C(\infty)} \geq \sup_{n\geq 2} \Gamma(|\Phi(a)|)(\frac{1}{n!})$, and for any $n \geq 2$,

$$\begin{aligned}\Gamma(|\Phi(a)|)(\frac{1}{n!}) &= n! \int_0^{\frac{1}{n!}} \Big| \sum_{k\geq n} a_k \mathbb{1}_{I_k}(t) \Big| dt \\ &\geq |a_n| \frac{n}{n+1} - n! \sum_{k\geq n+1} |a_k||I_k| \\ &\geq |a_n| \frac{n}{n+1} - \frac{\|a\|_\infty}{n+1}. \end{aligned}$$

We obtain that $\|\Phi(a)\|_{C(\infty)} \geq \dfrac{\|a\|_\infty}{2}$ and this finishes the proof. $\square$

Using the preceding proposition and the Müntz theorems in $\mathcal{C}$ and in L^p, we state a Müntz theorem for the Cesàro function spaces as follows:

Theorem 2.3. *Let $\Lambda = (\lambda_k)_{k=0}^\infty$ be an increasing sequence of nonnegative real numbers, $1 \leq p < +\infty$ (resp. $p = +\infty$). Then the following are equivalent:*

(i) The space $M(\Lambda) = span\{x^{\lambda_k} : k \in \mathbb{N}\}$ is dense in Ces_p (resp. the space $span\{1, x^{\lambda_0}, x^{\lambda_1}, \cdots\}$ is dense in the closure of $\mathcal{C}$ in Ces_∞).

(ii) The sequence Λ satisfies $\displaystyle\sum_{k\geq 1} \frac{1}{\lambda_k} = +\infty\cdot$

Moreover, if Λ *satisfies the Müntz condition* $\sum_{k=0}^{\infty} 1/\lambda_k < +\infty$*, the sequence* $(x^{\lambda_n})_n$ *is a minimal sequence in* Ces_p *for any* $p \in [1, +\infty]$*. In particular, for any* $\mu \in \mathbb{R}_+$ *such that* $\mu \notin \Lambda$*, we have* $\mathrm{dist}(x^\mu, M(\Lambda)) > 0$.

Proof. Assume that Λ satisfies $\sum_{k\geq 0} 1/\lambda_k = +\infty$ and fix a continuous function f on $[0,1]$. We first treat the case where $p = +\infty$. By the Müntz theorem on $\mathcal{C}$, there exists a sequence of polynomials $f_n \in \mathit{span}\{1, x^{\lambda_0}, x^{\lambda_1}, \cdots\}$ such that $\|f_n - f\|_\infty \to 0$ when $n \to +\infty$. Using the boundedness of the inclusion $\mathcal{C} \subset \mathrm{Ces}_p$, we get that $\|f_n - f\|_{C(p)} \to 0$ when $n \to +\infty$. Hence, the space $\mathit{span}\{1, x^{\lambda_0}, x^{\lambda_1}, \cdots\}$ is dense in the closure of the continuous functions in Ces_p. Now for the case where $p \in [1, +\infty)$, we take $q = p$ when $p > 1$ and any $q > 1$ when $p = 1$. By the Müntz theorem in L^q, we know that there exists a sequence $(f_n)_n \in M(\Lambda)$ such that $\|f_n - f\|_q \to 0$ when $n \to +\infty$. Hence, we compute

$$\|f_n - f\|_{C(p)} = \|\Gamma(|f_n - f|)\|_p \leq \|\Gamma(|f_n - f|)\|_q \leq q'\|f_n - f\|_q \to 0.$$

In the last inequality, we use the well-known Hardy inequality. Finally, by Proposition 2.2, we deduce the density. For the "only if" part, we consider a sequence Λ satisfying $\sum 1/\lambda_n < +\infty$ and we fix $\mu \in \mathbb{R}_+ \setminus \Lambda$. For any $a \in (0,1)$ and for any Müntz polynomial $f \in M(\Lambda)$, we write

$$\begin{aligned}\|x^\mu - f\|_{C(p)} &\geq (1-a)^{\frac{1}{p}} \|x^\mu - f\|_{L^1([0,a])} \\ &= a(1-a)^{\frac{1}{p}} \int_0^1 |(au)^\mu - f(au)| du \\ &\geq (1-a)^{\frac{1}{p}} a^{\mu+1} \inf_{g \in M(\Lambda)} \|x^\mu - g\|_1.\end{aligned}$$

According to the Müntz theorem in L^1, we have that $\inf_{g\in M(\Lambda)} \|x^\mu - g\|_1 > 0$. Hence $M(\Lambda)$ is not dense in Ces_p. ☐

Remark 2.4. Even if Λ satisfies the condition $\sum_{n\geq 1} 1/\lambda_n = +\infty$, we need to assume that $0 \in \Lambda$ in order to approximate the constant functions by Müntz polynomials in Ces_∞ because $\|1 - f\|_{C(\infty)} \geq |\Gamma(|1 - f|)(0)| = 1$ if $f \in \mathcal{C}_0$. But this problem does not happen in the spaces Ces_p when $p \in [1, +\infty)$. In other words, thanks to Proposition 2.2, the space $M(\Lambda)$ is dense in Ces_p even when $\lambda_0 > 0$.

Now, we can define the Müntz-Cesàro spaces as follows:

Definition 2.5. Let $\Lambda = (\lambda_n)_{n\geq 0} \subset \mathbb{R}_+$ be an increasing sequence satisfying *the Müntz condition*

$$\sum_{n\geq 1} \frac{1}{\lambda_n} < +\infty.$$

For $p \in [1, +\infty)$ (resp. $p = +\infty$), the classical *Müntz space* M_Λ^p (resp. M_Λ^∞) is defined as the closure of the space of *Müntz polynomials* $M(\Lambda)$ in L^p (resp. $\mathcal{C}$). In the same way, for $p \in [1, +\infty]$, we define the *Müntz-Cesàro space* $M_\Lambda^{\mathrm{Ces}_p}$ as the closure of $M(\Lambda)$ in Ces_p. By Theorem 2.3, it is a strict subspace of Ces_p. Concretely, in the sequel, we shall always assume that the inequality, called *gap-condition*,

$$\inf_{n\geq 0} \left(\lambda_{n+1} - \lambda_n\right) > 0$$

is fulfilled in order to work with the spaces of analytic functions (see Proposition 2.7 below).

Remark 2.6. The norms $\| \,.\, \|_{C(\infty)}$ and $\| \,.\, \|_1$ are equivalent on $M(\Lambda)$. Indeed, on the one hand we have $\|f\|_{C(\infty)} \geq \Gamma(|f|)(1) = \|f\|_1$, for any function $f \in \mathrm{Ces}_p$. On the other hand, by a bounded Bernstein-type inequality on M^1_Λ (see [BE, E.3 p. 178]) there exists a constant $C_{1/2} \in \mathbb{R}_+$ such that for any $f \in M(\Lambda)$ we have

$$\begin{aligned}\|f\|_{C(\infty)} &\leq \sup_{x\in[0,1/2]} \frac{1}{x}\int_0^x |f(t)|dt + \sup_{x\in(1/2,1]} \frac{1}{x}\int_0^x |f(t)|dt\\ &\leq \sup_{t\in[0,1/2]} |f(t)| + 2\int_0^1 |f(t)|dt\\ &\leq \big(C_{1/2}+2\big)\|f\|_1.\end{aligned}$$

Hence we get that $M^{\mathrm{Ces}_\infty}_\Lambda = M^1_\Lambda$, and the spaces have equivalent norms.

The next proposition is a version of the Clarkson-Erdös theorem (see [CE],[S]) for the Müntz-Cesàro spaces. It is indeed a consequence of the Clarkson-Erdös theorem in L^p and in $\mathcal{C}$.

Proposition 2.7. *Let $p \in [1, +\infty)$ (resp. $p = +\infty$) and let $\Lambda = (\lambda_k)_{k=0}^\infty$ be an increasing sequence of non-negative real numbers. Assume that Λ satisfies the Müntz and the gap conditions. For a function $f \in \mathrm{Ces}_p$ (resp. $f \in \overline{\mathcal{C}}^{\mathrm{Ces}_\infty}$), the following are equivalent:*

(i) $f \in M^{\mathrm{Ces}_p}_\Lambda$.

(ii) There exist $\widetilde{f} \in \mathrm{Ces}_p$, with $f = \widetilde{f}$ a.e. on $[0,1]$ and a sequence $\big(a_n\big)$ of complex numbers, such that

$$\forall x \in [0,1), \qquad \widetilde{f}(x) = \sum_{n=0}^{\infty} a_n x^{\lambda_n}.$$

Proof. The case of $M^{\mathrm{Ces}_\infty}_\Lambda$ is actually free by the Clarkson-Erdös theorem in M^1_Λ. Nevertheless we see below that the proof for $M^{\mathrm{Ces}_p}_\Lambda$ also holds for $M^{\mathrm{Ces}_\infty}_\Lambda$. For the part $(i) \Rightarrow (ii)$, we consider a sequence of Müntz polynomials $(f_n)_n \in M(\Lambda)$ which tends to f in $C(p)$ when $n \to +\infty$. By the Hardy inequality, we have

$$\|\Gamma(f_n) - \Gamma(f)\|_p \leq \|\Gamma(|f_n - f|)\|_p = \|f_n - f\|_{C(p)}.$$

Since $\Gamma(f)$ is the limit in L^p (resp. in $\mathcal{C}$) of a sequence of Müntz polynomials, we have that $\Gamma(f) \in M^p_\Lambda$. By the Clarkson-Erdös theorem in L^p (resp. in $\mathcal{C}$) (see for instance [BE, E.1 p. 311]), we know that there exists a sequence $(b_n) \in \mathbb{C}$ which satisfies $\limsup |b_n|^{\frac{1}{\lambda_n}} \leq 1$, and that

$$\forall x \in [0,1), \;\; \Gamma(f)(x) = \sum_n b_n x^{\lambda_n}.$$

Now, we define the function $\widetilde{f}$ by $\widetilde{f}(x) = \sum_n b_n(\lambda_n + 1)x^{\lambda_n}$. Clearly, this series converges uniformly on the compacts subsets of $[0,1)$ because it has the same radius of convergence as $\Gamma(f)$. Moreover, we have that $\Gamma(\widetilde{f})(x) = \Gamma(f)(x)$ for any $x \in [0,1)$ which gives that $\widetilde{f} = f$ almost everywhere. To prove that $(ii) \Rightarrow (i)$, we follow the same lines as in [GL, Cor. 6.2.4]. For this, we let $f \in \mathrm{Ces}_p$ (resp. $f \in \overline{\mathcal{C}}^{\mathrm{Ces}_\infty}$) to be a function that satisfies $f(x) = \sum_{n=0}^{\infty} a_n x^{\lambda_n}$ for $x \in [0,1)$. As the series converges for any $x \in [0,1)$, we have $\limsup |a_n|^{\frac{1}{\lambda_n}} \leq 1$. Given a function h on $[0,1)$ and $\rho \in (0,1)$, we will denote by h_ρ the function defined by $h_\rho(t) = h(\rho t)$. For the sequence of partial sums $(f_m) \in M(\Lambda)$ given by $f_m(t) = \sum_{n=0}^{m} a_n t^{\lambda_n}$, we define the corresponding functions $(f_m)_\rho$ and compute

$$\|f_\rho - (f_m)_\rho\|_{C(p)} \leq \sum_{n=m+1}^{+\infty} \frac{|a_n|\rho^{\lambda_n}}{\lambda_n + 1} \underset{m\to+\infty}{\longrightarrow} 0.$$

Therefore, $f_\rho \in M_\Lambda^{\mathrm{Ces}_p}$ for any $\rho \in (0,1)$. Now, we claim that $\lim_{\rho\to 1} \|f - f_\rho\|_{C(p)} = 0$ which would give that $f \in M_\Lambda^{\mathrm{Ces}_p}$ and finish the proof of the proposition. To check this, we let $\varepsilon > 0$ and consider a continuous function g satisfying $\|f - g\|_{C(p)} < \varepsilon$, when p is finite, and by assumption when $p = +\infty$. The existence of such a function is assured by Proposition 2.2. Now, for any $\rho \in (0,1)$ and $h \in \mathrm{Ces}_p$, the estimate $\|h_\rho\|_{C(p)} \leq \frac{1}{\rho}\|h\|_{C(p)}$ gives

$$\begin{aligned} \|f - f_\rho\|_{C(p)} &\leq \|f_\rho - g_\rho\|_{C(p)} + \|g - g_\rho\|_{C(p)} + \|f - g\|_{C(p)} \\ &\leq \Big(\frac{1}{\rho} + 1\Big)\|f - g\|_{C(p)} + \|g - g_\rho\|_\infty \quad . \end{aligned}$$

The uniform continuity of g on $[0,1]$ implies $\lim_{\rho\to 1} \|g - g_\rho\|_\infty = 0$ and we obtain as claimed that $\|f - f_\rho\|_{C(p)} \leq 2\varepsilon$, and hence $f \in M_\Lambda^{\mathrm{Ces}_p}$. □

The following estimate is a bounded Bernstein-type inequality. To establish such an estimate, we will use the analogue well known inequality in the classical Müntz spaces.

Proposition 2.8. *Let $p \in [1,+\infty]$ and $\Lambda = (\lambda_n)_n$ be a sequence of non-negative numbers satisfying the Müntz and gap conditions. Then for every $\varepsilon \in (0,1)$, there exists a constant $c(\varepsilon,\Lambda)$ depending only on ε and Λ such that*

$$\|f'\|_{[0,1-\varepsilon]} \leq c(\varepsilon,\Lambda)\|f\|_{\mathrm{Ces}_p}$$

for every Müntz polynomial $f \in span\left\{x^{\lambda_0}, x^{\lambda_1}, ...\right\}$.

Proof. Let $\varepsilon \in (0,1)$ and fix the two real numbers $a, \eta \in (0,1)$ such that $a(1-\eta) > 1-\varepsilon$. For any Müntz polynomial $f \in M(\Lambda)$, we know from the bounded Bernstein inequality in M_Λ^1 (see [BE, E.3 p. 178]) that there exists a constant $C_\eta \in \mathbb{R}_+$ that does not depend on f and a satisfying

$$\|(f_a)'\|_{[0,1-\eta]} \leq C_\eta \|f_a\|_1,$$

where f_a is the function defined as in the proof of Proposition 2.7. Now, we compute

$$\begin{aligned} \|f\|_{\mathrm{Ces}_p} &\geq (1-a)^{\frac{1}{p}}\|f\|_{L^1([0,a])} \\ &= a(1-a)^{\frac{1}{p}}\|f_a\|_1 \\ &\geq \frac{a(1-a)^{\frac{1}{p}}}{C_\eta}\|(f_a)'\|_{[0,1-\eta]}. \end{aligned}$$

By the choice of a and η above, we obtain the result with $c(\varepsilon,\Lambda) = \frac{C_\eta}{a(1-a)^{\frac{1}{p}}}$. ☐

We finish this section with this last useful result. The proof follows the same lines as in [AL, Cor. 2.5].

Corollary 2.9. *Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be an increasing sequence of non-negative real numbers satisfying the Müntz and gap conditions. For any bounded sequence $(f_n)_{n=1}^\infty \in M_\Lambda^{\mathrm{Ces}_p}$, there exist $f \in \mathrm{Ces}_p$ and a subsequence $(f_{n_k})_{k=1}^\infty$ that converges to f uniformly on every compact subset of* $[0,1)$.

Proof. Assume that $(f_n)_{n\geq 1}$ is a bounded sequence in $M_\Lambda^{\mathrm{Ces}_p}$ and fix $\varepsilon > 0$. If $\lambda_0 > 0$ then $M(\Lambda) \subset \mathcal{C}_0$ and from Proposition 2.8 and the mean value theorem, the sequence $(f_n)_n$ is bounded and equicontinuous on the compact interval $[0, 1-\varepsilon]$. If $\lambda_0 = 0$, then by the Müntz theorem in Ces_p (Theorem 2.3), there exists $\delta > 0$ such that

$$\inf_{g\in M(\Lambda\setminus\lambda_0)} \|t^{\lambda_0} - g\|_{C(p)} \geq \delta,$$

since $\Lambda \setminus \lambda_0$ also satisfies the Müntz condition. Now, we write $f_n = f_n(0)t^{\lambda_0} + g$ with $g \in M(\Lambda \setminus \lambda_0)$ and the previous estimate gives $|f_n(0)| \leq \frac{1}{\delta}\|f_n\|_{C(p)}$. In this case, we obtain that $(f_n)_n$ is bounded and equicontinuous on $[0, 1-\varepsilon]$. Using the Arzela-Ascoli theorem, we know that there exists an extraction $(n_k)_k$ such that (f_{n_k}) converges uniformly on $[0, 1-\varepsilon]$. By induction, we can construct a sequence of infinite sets $(S_j)_{j\geq 1}$ of integers with $\mathbb{N} \supset S_1 \supset S_2 \supset \cdots$ such that $(f_n)_{n\in S_j}$ converges uniformly on $[0, 1-\frac{1}{j}]$. By applying a diagonal method, we obtain an infinite set S such that $(f_n)_n$ converges uniformly on every compact subset of $[0,1)$ to a measurable function f when $n \to +\infty$ and $n \in S$. Finally, Fatou's lemma (twice if $p < +\infty$) allows to obtain that $\|f\|_{C(p)} \leq \sup_n \|f_n\|_{C(p)}$ which finishes the proof. ☐

3 General Lemmas for the essential norm

In this section, we will give some general criteria to compute the essential norm. We first recall a result from [AHLM].

Definition 3.1. We say that a sequence $\big(\tilde{x}_m\big)_{m\in\mathbb{N}}$ is a *block-subsequence* of $\big(x_n\big)_n$ if there is a sequence of non empty finite subsets of integers $\big(I_m\big)_{m\in\mathbb{N}}$ with $\max I_m < \min I_{m+1}$, and $c_i \in [0,1]$ such that for all $m \in \mathbb{N}$,

$$\sum_{j\in I_m} c_j = 1 \quad \text{and} \quad \tilde{x}_m = \sum_{j\in I_m} c_j x_j.$$

Lemma 3.2. [AHLM, Lemma 3.1] *Let X, Y be two Banach spaces, and $T : X \to Y$ be a bounded operator. Let $\big(x_n\big)_{n\in\mathbb{N}}$ be a normalized sequence in X and $\alpha > 0$.*

- *(i) Assume that for any subsequence $\big(x_{\varphi(n)}\big)_{n\in\mathbb{N}}$ and any $g \in Y$, we have $\limsup_{n\to+\infty} \big\|T\big(x_{\varphi(n)}\big) - g\big\| \geq \alpha$. Then $\|T\|_e \geq \alpha$.*

(ii) Assume that for any block-subsequence $\big(\tilde{x}_n\big)_{n\in\mathbb{N}}$ *and any* $g \in Y$*, we have* $\limsup\limits_{n\to+\infty} \big\|T\big(\tilde{x}_n\big) - g\big\| \geq \alpha$. *Then* $||T||_{e,w} \geq \alpha$.

Definition 3.3. Let (X, d) be a metric space and $\alpha \in \mathbb{R}_+$. We say that a sequence $(x_n)_n \in X$ is *α-separated* if $d(x_n, x_m) \geq \alpha$ for all $n \neq m$.

The following lemma is a consequence of Lemma 3.2. It will be used to find a lower estimate for the essential norm for some operators.

Lemma 3.4. *Let* X, Y *be two Banach spaces,* $T : X \longrightarrow Y$ *a linear operator and* $\alpha \in \mathbb{R}_+$.

(i) If the range of the unit ball $T(B_X)$ *contains an* α*-separated sequence, then* $\|T\|_e \geq \dfrac{\alpha}{2}\cdot$

(ii) If the range of the unit ball $T(B_X)$ *contains a sequence* $(y_n)_n$ *such that any block-subsequence* $(\widetilde{y}_m)_m$ *is* α*-separated, then* $\|T\|_{e,w} \geq \dfrac{\alpha}{2}\cdot$

Proof. We prove only (ii) since (i) is similar (and actually easier). Let $(y_n)_n \in T(B_X)$ such that any block-subsequence $(\widetilde{y}_m)_m$ of $(y_n)_n$ is α-separated in Y. Fix $g \in Y$, as $n \neq m \in \mathbb{N}$, we have

$$\alpha \leq \|\widetilde{y}_n - \widetilde{y}_m\| \leq \|\widetilde{y}_n - g\| + \|\widetilde{y}_m - g\|.$$

Therefore, there is at most one integer $n \in \mathbb{N}$ such that $\|\widetilde{y}_n - g\| < \dfrac{\alpha}{2}$, which yields $\limsup\limits_{n\to\infty} ||T(x_n) - g|| \geq \dfrac{\alpha}{2}\cdot$ The result follows by Lemma 3.2 (ii). ☐

The following example shows that the lower estimate in Lemma 3.4 can be sharp.

Example 3.5. The sequential Volterra operator $v : \ell^1 \longrightarrow c$ is defined by $v(x) = \Big(\sum\limits_{k=0}^{n} x_k\Big)_n$ for any $x = (x_k)_k \in \ell^1$. We have $\|v\|_e = \dfrac{1}{2}\cdot$

Proof. We consider $(e_n)_n$ the canonical basis of ℓ^1 and for $n \in \mathbb{N}$ we denote by $f_n := v(e_n)$. For a given $n \in \mathbb{N}$, we have $f_n = (f_{n,k})_k \in v(B_{\ell^1})$, where $f_{n,k} = 0$ if $k < n$ and $f_{n,k} = 1$ if $k \geq n$. Since $(f_n)_n$ is 1-separated in c, Lemma 3.4 gives the lower bound. For the upper bound, we consider the rank-one operator $K : \ell^1 \to c$, defined by $K = \frac{1}{2}\mathbb{1} \otimes \mathrm{Tr}$, where $\mathbb{1}$ is the constant sequence equal to 1 and $\mathrm{Tr} \in (\ell^1)^*$ is the trace functional. For any $x \in \ell^1$, we have

$$\|(v - K)(x)\|_c = \sup_n \left|\sum_{k=0}^{n} x_k - \sum_{k=0}^{+\infty} \frac{x_k}{2}\right| = \frac{1}{2} \sup_n \left|\sum_{k=0}^{n} x_k - \sum_{k=n+1}^{+\infty} x_k\right| \leq \frac{\|x\|_{\ell^1}}{2}\cdot$$

Since K is compact, we get that $\|v\|_e \leq \|v - K\| \leq \frac{1}{2}$. This finishes the proof. ☐

Definition 3.6. Let X be a Banach space. We say that $P : X \to \mathbb{R}_+$ is a *subnorm* on X if P satisfies

(i) $\forall x, y \in X$, $P(x + y) \leq P(x) + P(y)$ (triangle inequality) ;

(ii) $\forall x \in X$, $P(x) \leq ||x||$.

Lemma 3.7. *Let* $\alpha \in \mathbb{R}^+$*. Let* X, Y *be two Banach spaces, and* $T : X \longrightarrow Y$ *a linear operator. Let* $(P_k)_{k\in\mathbb{N}}$ *be a family of subnorms on* Y*. Assume that:*

(i) For any $g \in Y$, $\inf\limits_{k\in\mathbb{N}} P_k(g) = 0$.

(ii) There exists a sequence $(h_n)_n \in B_X$ *such that*

$$\forall k \in \mathbb{N}, \qquad \liminf_{n\to+\infty} P_k(T(h_n)) \geq \alpha.$$

Then $||T||_e \geq \alpha$.

Proof. Let $S : X \longrightarrow Y$ be a compact operator, and $\varepsilon > 0$. From the defnition of S, there exists an extraction $(n_j)_j$ in $\mathbb{N}$ such that $S(h_{n_j}) \to g \in Y$. As $\inf_{k\in\mathbb{N}} P_k(g) = 0$, we set $k_0 \in \mathbb{N}$ such that $P_{k_0}(g) \leq \varepsilon$. Now, we set $j_0 \in \mathbb{N}$ in a way that we have simultaneously $\|S(h_{n_j}) - g\| \leq \varepsilon$ and $P_{k_0}(T(h_{n_j})) \geq \alpha - \varepsilon$ for any $j \geq j_0$. For $j = j_0$, we have

$$\begin{aligned}
\|T\|_e \geq \|T - S\| &\geq \|T(h_{n_j}) - S(h_{n_j})\| \\
&\geq \|T(h_{n_j}) - g\| - \|S(h_{n_j}) - g\| \\
&\geq P_{k_0}(T(h_{n_j}) - g) - \|S(h_{n_j}) - g\| \\
&\geq P_{k_0}(T(h_{n_j})) - P_{k_0}(g) - \varepsilon \\
&\geq \alpha - 3\varepsilon.
\end{aligned}$$

As this holds for all $\varepsilon > 0$, we thus get that $\|T\|_e \geq \alpha$. ☐

In the following, we present a variant of the preceding lemma (it is actually a direct consequence).

Lemma 3.8. *Let* $\alpha \in \mathbb{R}^+$*. Let* X, Y *be two Banach spaces, and* $T : X \longrightarrow Y$ *be a linear operator. Let* $(P_k)_{k\in\mathbb{N}}$ *be a family of subnorms on* Y*. Assume that:*

(i) For any $g \in Y$, $\lim\limits_{k\to+\infty} P_k(g) = 0$.

(ii) There exists a sequence $(h_n)_n \in B_X$ *such that*

$$\forall k \in \mathbb{N}, \qquad \limsup_{n\to+\infty} P_k(T(h_n)) \geq \alpha.$$

Then $||T||_e \geq \alpha$.

Proof. By hypothesis, for every $k \geq 1$, there exists an extraction $\theta_k : \mathbb{N} \to \mathbb{N}$ such that for any $n \in \mathbb{N}$ such that $n \geq k$,

$$P_k\big(T\big(h_{\theta_k(n)}\big)\big) \geq \alpha - \frac{1}{k}\cdot$$

By induction, we can also assume that $(\theta_k(k))_k$ is increasing. Now consider the subnorms $\widetilde{P_k} = \sup_{m\geq k} P_m$ and the sequence $h'_n = \big(h_{\theta_n(n)}\big)$ in the unit ball of X. On the one hand, we have that $\inf \widetilde{P_k}(g) = \limsup P_k(g) = 0$ for any g. On the other hand, we write for any k

$$\liminf_{n\to+\infty} \widetilde{P_k}(T(h_n)) \geq \liminf_{n\to+\infty} P_n(T(h'_n)) \geq \alpha.$$

Finally, Lemma 3.7 gives the conclusion. ☐

The following corollary will be particularly efficient when Y is an L^p space.

Corollary 3.9. *Let (Ω, μ) be a measure space, X be a Banach space and $T : X \longrightarrow L^p(\Omega, \mu)$ be a linear operator. Assume that there exist a decreasing sequence of measurable subsets $(A_k)_k$ of Ω, a sequence $(h_n)_n$ in B_X, and a number $\alpha > 0$ such that:*

(i) The sequence of Borel sets (A_k) satisfies $\mu\left(\bigcap_k A_k\right) = 0$.

(ii) For any $k \in \mathbb{N}$, we have $\limsup_{n\to+\infty} \left(\int_{A_k} |T(h_n)|^p d\mu\right)^{\frac{1}{p}} \geq \alpha$.

Then $||T||_e \geq \alpha$.

Proof. According to the monotone convergence theorem, the sequence of subnorms $P_k : L^p(\Omega, \mu) \longrightarrow \mathbb{R}_+$ defined by $P_k(f) := ||f||_{L^p(A_k,\mu)}$, converges pointwise to 0 on $L^p(\mu)$. Then Lemma 3.8 gives the result. □

Definition 3.10. Let (E, d), (E', d') be two metric spaces, and $f : E \longrightarrow E'$ be a function. For $\alpha \in \mathbb{R}^+$ and $a \in E$, we say that f has *a jump at the point a with height at least α* if

$$\forall r > 0,\ \delta(f(B(a, r))) \geq \alpha,$$

where $\delta(A)$ denotes the diameter of $A \subset E'$.

The following result can be found in [AHLM]. Here we show that it is a particular case of Lemma 3.7.

Corollary 3.11. *Let X be a Banach space, K be a metric compact space, and $T : X \longrightarrow \mathcal{C}(K)$ be an operator. Assume that there exist a sequence $(h_n)_n$ in B_X, an element $a \in K$ and a function $g : K \to \mathbb{C}$ such that:*

(i) $(T(h_n))_n$ converges pointwise to g.

(ii) g has a jump at the point a with height 2α.

Then $||T||_e \geq \alpha$.

Proof. We apply Lemma 3.7 for the sequence of subnorms $(P_k)_k$ on $\mathcal{C}$ defined by

$$P_k(f) := \frac{1}{2}\delta\left(f\left(B(a, \frac{1}{k})\right)\right).$$

One can easily check that the assumptions of the lemma are satisfied. □

Corollary 3.12. *Let $T : E \longrightarrow c$ be a linear operator, and $\alpha \in \mathbb{R}^+$. Assume that there is a sequence $(f_n)_n = \left((f_{n,j})_{j\in\mathbb{N}}\right)_n$ in $T(B_E)$, such that for all $k \in \mathbb{N}$, $\limsup_{n\to\infty} \delta(\{f_{n,j}, j \geq k\}) \geq 2\alpha$. Then $||T||_e \geq \alpha$.*

Proof. We apply Lemma 3.8 for the sequence of subnorms $(P_k)_k$ on c defined by

$$P_k((x_n)_n) := \frac{1}{2}\delta(\{x_i : i \geq k\}).$$

We can also use Corollary 3.11, by seeing c as a $\mathcal{C}(K)$-space where $K = \mathbb{N} \cup \{\infty\}$. □

The following lemma is a natural generalisation of [CFT, Lemma 3.4] for all $p \in [1, +\infty)$, and the proof can be easily adapted. However, we can also see this result as a consequence of Lemma 3.7.

Lemma 3.13. *Let* (Ω, μ) *be a measure space,* X *be a Banach space and* $T : X \to L^p(\mu)$ *be a bounded operator. For a decreasing sequence of measurable subset* $A_n \subset \Omega$ *satisfying* $\mu(\bigcap_n A_n) = 0$*, the sequence* $(R_n)_n$ *of projection operators is defined by*

$$R_n : \left\{ \begin{array}{ccc} L^p(\mu) & \to & L^p(A_n, \mu) \\ f & \mapsto & f\mathbb{1}_{A_n}. \end{array} \right.$$

If for any $n \in \mathbb{N}$*, the operator* $T - R_nT$ *is compact, then the essential norm of* T *is given by*

$$\|T\|_e = \lim_{n \to +\infty} \|R_nT\|.$$

Proof. Since for any $x \in X$, $(\|R_nT(x)\|)_n$ is a decreasing sequence, we get that $(\|R_nT\|)_n$ converges to a number $\alpha \in \mathbb{R}_+$ when $n \to +\infty$. By the compactness of $(T - R_nT)$, we clearly have $\|T\|_e \leq \|R_nT\|$ for any n and hence $\|T\|_e \leq \alpha$. For the lower bound, we fix a sequence $h_n \in B_X$ which satisfies

$$\|R_nT(h_n)\|_p \geq \|R_nT\| - \frac{1}{n}\cdot$$

For $k \leq n$, we have $\|R_kT(h_n)\|_p = \|T(h_n)\|_{L^p(A_k,\mu)} \geq \|R_nT(h_n)\|_p$. Hence we apply Corollary 3.9 to the sequences $(A_k)_k$ and $(h_n)_n$ to get $\|T\|_e \geq \alpha$. ☐

The following proposition shows that for some classes of operators, the essential norm depends only on the range of the unit ball.

Proposition 3.14. *Let* X, Y *be two Banach spaces such that* Y *has a Schauder basis* $(e_n)_n$*. We consider the natural projections* $P_N : Y \to \mathrm{span}\,\{e_k; 0 \leq k \leq N\}$ *defined by* $P_N\left(\sum_{k=0}^{\infty} x_ke_k\right) = \sum_{k=0}^{N} x_ke_k$*, and* $R_N = I - P_N$*. Assume that* $\|R_N\| \leq 1$*, for any* $N \in \mathbb{N}$*. Then, for two bounded operators* $T, T' : X \to Y$ *with* $T(B_X) = T'(B_X)$*, we have*

$$\|T\|_e = \|T'\|_e = \lim_{N \to +\infty} \|R_NT\|.$$

Proof. Since P_NT has finite rank, it is compact for each N and we clearly have $\|T\|_e \leq \|T - P_NT\| = \|R_NT\|$ for all N. Hence $\|T\|_e \leq \liminf\limits_{N \to +\infty} \|R_NT\|$. To get the lower estimate, we let $\varepsilon > 0$ and $S : X \to Y$ to be a compact operator. We take a sequence $(x_N)_N \subset X$ such that $\|x_N\| = 1$ and $\|R_NTx_N\| \geq \|R_NT\| - \varepsilon$. The sequence $(Sx_N)_N$ contains a convergent subsequence, say $Sx_{N_j} \to y \in Y$. We have

$$\|(T - S)x_{N_j}\| \geq \|Tx_{N_j} - y\| - \|Sx_{N_j} - y\|$$

whence,

$$\limsup_j \|(T - S)x_{N_j}\| \geq \limsup_j \|Tx_{N_j} - y\|\cdot$$

Since $y \in Y$, there exists a positive integer n_0 such that $\|R_Ny\| \leq \varepsilon$ for all $N \geq n_0$. Then, if $N_j \geq n_0$, we have

$$\|Tx_{N_j} - y\| \geq \|R_{N_j}(Tx_{N_j} - y)\| \geq \|R_{N_j}Tx_{N_j}\| - \varepsilon \geq \|R_{N_j}T\| - 2\varepsilon.$$

It follows that

$$\|T-S\| \geq \limsup_{N\to\infty} \|R_N T\| - 2\varepsilon.$$

As this is true for any compact S, we get

$$\|T\|_e \geq \limsup_{N\to\infty} \|R_N T\| - 2\varepsilon.$$

This yields to the desired inequality as ε is chosen arbitrary. $\square$

The following result is an analogue estimate of the previous results that gives a lower bound of the distance between an operator T with values in $L^1(\mu)$ to the space of weakly compact operators.

Proposition 3.15. *Let (Ω,μ) be a measure space, X be a Banach space and $T : X \longrightarrow L^1(\Omega,\mu)$ be a linear operator. Assume that there exist a number $\alpha > 0$, a sequence $(h_n)_n$ in the unit ball of X and a sequence of measurable sets $(A_n)_n$ in Ω with:*

(i) $\mu(A_k) \to 0$ when $k \to +\infty$.

(ii) For any $k \in \mathbb{N}$, $\limsup_{n\to+\infty} \Big(\int_{A_k} |T(h_n)| d\mu \Big) \geq \alpha$.

Then, we have $\|T\|_{e,w} \geq \alpha$.

Proof. Following the ideas of the proof of Lemma 3.8, we may assume, without loss of generality, that for any $k \in \mathbb{N}$, $\liminf_{n\to+\infty} \Big(\int_{A_k} |T(h_n)| d\mu \Big) \geq \alpha$. Let $S : X \to L^1(\Omega,\mu)$ be a weakly compact operator. Since the set $H = \{S(h_n), n \in \mathbb{N}\}$ is bounded and relatively weakly compact, then it is uniformly integrable [Wo, p.137]. That means that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\mu(B) \leq \delta_\varepsilon \Rightarrow \int_B |S(h_n)| d\mu \leq \varepsilon, \quad \forall n \in \mathbb{N}.$$

But, for any $\varepsilon > 0$ there exists k such that $\mu(A_k) < \delta_\varepsilon$. Therefore we compute

$$\begin{aligned}
\|T-S\| &\geq \|(T-S)(h_n)\|_{L^1(\mu)} \\
&\geq \int_{A_k} |Th_n - Sh_n| d\mu \\
&\geq \int_{A_k} |Th_n| d\mu - \int_{A_k} |Sh_n| d\mu \\
&\geq \alpha - \varepsilon.
\end{aligned}$$

Finally, we deduce $\|T\|_{e,w} \geq \alpha$ which is the desired result. $\square$

4 Essential norm of some Cesàro operators

In this section, we will compute the essential norm the Cesàro operator (discrete and continuous) defined between different Banach spaces.

4.1 Cesàro operators on Lebesgue spaces

In the following, we shall compute the essential norm of the continuous Cesàro operator $\Gamma_p : L^p \to L^p$ and of the discrete Cesàro operator $\gamma_p : \ell^p \to \ell^p$ for $p \in (1, +\infty]$. We shall distinguish the case p finite and $p = +\infty$. Of course, the key information in the two following theorems concerns the essential norm. The value of the operator norm of the continuous and discrete Cesàro operators is a well known fact.

Theorem 4.1. *Let $p \in (1, +\infty)$ and $\Gamma_p : L^p \to L^p$ be the continuous Cesàro operator. Then,*

$$\|\Gamma_p\|_e = \|\Gamma_p\| = p',$$

where $p' = \frac{p}{p-1}$. In particular, we have $a_n(\Gamma_p) = p'$ for any $n \in \mathbb{N}$.

Proof. Using the Hardy inequality (see [HLP, Th.327 p.240]), we have the upper bound

$$\|\Gamma_p\|_e \leq \|\Gamma_p\| \leq p' \quad .$$

To prove the lower bound, we apply Corollary 3.9 to any sequence of subsets $A_k = [0, \delta_k]$ (where δ_k is a decreasing sequence converging to 0) and to the sequence $(h_n)_n$ defined by $h_n(x) = (p\varepsilon_n)^{\frac{1}{p}} x^{-\frac{1}{p}+\varepsilon_n} \in B_{L^p}$ where $\varepsilon_n \to 0$ as $n \to \infty$. By a straightforward computation, we have for any integer $k \in \mathbb{N}$,

$$\|\Gamma_p(h_n)\|^p_{L^p(A_k)} = \int_0^{\delta_k} \left(\frac{1}{x} \int_0^x (p\varepsilon_n)^{\frac{1}{p}} t^{-\frac{1}{p}+\varepsilon_n} dt \right)^p dx = \frac{(\delta_k)^{p\varepsilon_n}}{(\frac{1}{p'} + \varepsilon_n)^p}.$$

Therefore for any fixed k, we get $\liminf\limits_{n\to\infty} \|\Gamma_p(h_n)\|_{L^p(A_k)} \geq p'$. This finishes the proof. ☐

Theorem 4.2. *Let $p \in (1, +\infty)$, and $\gamma_p : \ell^p \to \ell^p$ be the discrete Cesàro operator. Then we have*

$$\|\gamma_p\|_e = \|\gamma_p\| = p',$$

where $p' = \frac{p}{p-1}\cdot$ In particular, we have $a_n(\gamma_p) = p'$ for any $n \in \mathbb{N}$.

Proof. Using the Hardy inequality (see [HLP, Th. 326 p 239]), we have the upper estimate

$$\|\gamma_p\|_e \leq \|\gamma_p\| \leq p'.$$

Now we prove the lower estimate. For $\varepsilon > 0$ and $N \in \mathbb{N}$, we consider the sequence $a^{(N)} = (a_n^{(N)})_n \in \ell^p$ with

$$a_n^{(N)} = \frac{(p\varepsilon)^{\frac{1}{p}} N^\varepsilon}{n^{\frac{1}{p}+\varepsilon}} \mathbb{1}_{[N,+\infty)}(n), \quad n \in \mathbb{N}.$$

The norm of $a^{(N)}$ is estimated as:

$$\|a^{(N)}\|_p^p = p\varepsilon N^{p\varepsilon} \sum_{n=N}^{\infty} \frac{1}{n^{1+p\varepsilon}} \underset{N\to+\infty}{\sim} p\varepsilon N^{p\varepsilon} \int_N^\infty \frac{1}{x^{1+p\varepsilon}} dx = 1.$$

By a simple computation, we have

$$\left(\gamma_p(a^{(N)})\right)_n = \begin{cases} 0 & \text{if } n < N \\ \dfrac{(p\varepsilon N^{p\varepsilon})^{\frac{1}{p}}}{n+1} \displaystyle\sum_{k=N}^{n} \frac{1}{k^{\frac{1}{p}+\varepsilon}} & \text{if } n \geq N. \end{cases}$$

Using the inequality between the Riemann series and the integral, we get

$$\sum_{k=a}^{b} \frac{1}{k^{\beta}} \geq \int_{a}^{b+1} \frac{1}{x^{\beta}} dx = \frac{(b+1)^{1-\beta} - a^{1-\beta}}{1-\beta}, \tag{3}$$

for any integer numbers a, b and any real $\beta \neq 1$. We estimate the norm of $\gamma_p(a^{(N)})$ for $\varepsilon \in (0, 1/p')$ as

$$\begin{aligned} \|\gamma_p(a^{(N)})\|_p^p &= \sum_{n=N}^{+\infty} \frac{p\varepsilon N^{p\varepsilon}}{(n+1)^p} \left(\sum_{k=N}^{n} \frac{1}{k^{\frac{1}{p}+\varepsilon}} \right)^p \\ &\geq \sum_{n=N}^{+\infty} \frac{p\varepsilon N^{p\varepsilon}}{(n+1)^p} \left(\frac{(n+1)^{\frac{1}{p'}-\varepsilon} - N^{\frac{1}{p'}-\varepsilon}}{\frac{1}{p'} - \varepsilon} \right)^p \\ &= \sum_{n=N}^{+\infty} \frac{p\varepsilon N^{p\varepsilon}}{(n+1)^{1+p\varepsilon}} \frac{(p')^p}{(1-p'\varepsilon)^p} \left(1 - \left(\frac{N}{n+1} \right)^{\frac{1}{p'}-\varepsilon} \right)^p \\ &\geq \sum_{n=N}^{+\infty} \frac{p\varepsilon N^{p\varepsilon}}{(n+1)^{1+p\varepsilon}} \frac{(p')^p}{(1-p'\varepsilon)^p} \left(1 - p \left(\frac{N}{n+1} \right)^{\frac{1}{p'}-\varepsilon} \right) \\ &= \frac{(p')^p}{(1-p'\varepsilon)^p} \left(\sum_{n=N}^{+\infty} \frac{p\varepsilon N^{p\varepsilon}}{(n+1)^{1+p\varepsilon}} - \sum_{n=N}^{+\infty} \frac{p^2\varepsilon N^{p\varepsilon+\frac{1}{p'}-\varepsilon}}{(n+1)^{1+p\varepsilon+\frac{1}{p'}-\varepsilon}} \right). \end{aligned}$$

Comparing again with an integral (like in (3)), the last estimate reduces to

$$\|\gamma_p(a^{(N)})\|_p^p \geq \frac{p'^p}{(1-p'\varepsilon)^p} \left(\left(\frac{N}{N+1} \right)^{p\varepsilon} - \frac{p^2\varepsilon}{p\varepsilon + \frac{1}{p'} - \varepsilon} \left(\frac{N}{N+1} \right)^{p\varepsilon + \frac{1}{p'} - \varepsilon} \right).$$

Letting $N \to \infty$ and as $\varepsilon \to 0$, we get $\lim\limits_{N\to\infty} \|\gamma_p(a^{(N)})\|_p \geq p'$. Now we apply Corollary 3.9 to the sets $A_k = \mathbb{N} \cap [k, +\infty)$ and to the sequence $(h_N) = (a^{(N)})$. Indeed, we clearly have that $\bigcap A_k = \emptyset$. Moreover, for a fixed $k \in \mathbb{N}$, we have

$$\|\gamma_p(a^{(N)})\|_{\ell^p(A_k)} = \|\gamma_p(a^{(N)})\|_p$$

when $N \geq k$ because the support of $\gamma_p(a^{(N)})$ is included in $\mathbb{N} \cap [N, +\infty)$. Hence, the essential norm of γ_p is equal to p'. □

Theorem 4.3. *Let $\Gamma_\infty : L^\infty \longrightarrow L^\infty$ be the Cesàro function operator, we then have*

$$\|\Gamma_\infty\|_{e,w} = \|\Gamma_\infty\|_e = \|\Gamma_\infty\| = 1.$$

In particular, we get that $a_n(\Gamma_\infty) = 1$ for any $n \in \mathbb{N}$.

Proof. First it is clear that $\|\Gamma_\infty\|_{e,w} \leq \|\Gamma_\infty\|_e \leq \|\Gamma_\infty\| = 1$. To prove the lower bound, we will fix $\varepsilon \in (0,1)$ and will define a sequence of functions $(h_n)_n$ in the unit ball of L^∞, such that any block-subsequence of $\Gamma_\infty(h_n)$ will be $(2-2\varepsilon)$-separated in L^∞. Therefore Lemma 3.4 (ii) will yield $\|\Gamma_\infty\|_{e,w} \geq 1-\varepsilon$, which will give the result. For this, we consider the sequence $(h_n)_n \in B_{L^\infty}$ by the following

$$h_n(x) = \begin{cases} -1 & \text{if } x \leq \varepsilon^n \\ 1 & \text{if } x > \varepsilon^n. \end{cases}$$

The sequence $H_n := \Gamma_\infty(h_n)$ satisfies $H_n = -1$ on $[0, \varepsilon^n]$ and $H_n(x) = \dfrac{x - 2\varepsilon^n}{x}$ if $x > \varepsilon^n$. Let $(\widetilde{H}_m)_m$ be a block-subsequence of $(H_n)_n$ defined by $\widetilde{H}_m = \sum\limits_{j \in I_m} c_j H_j$ (see Definition 3.1). For two integers k, l with $l > k$, we have

$$H_l(\varepsilon^k) = 1 - 2\frac{\varepsilon^l}{\varepsilon^k} \geq 1 - 2\varepsilon.$$

For any two integers numbers m, n with $m < n$, we set $k = \max I_m$ and compute

$$\begin{aligned} \|\widetilde{H}_n - \widetilde{H}_m\|_\infty &\geq |\widetilde{H}_n(\varepsilon^k) - \widetilde{H}_m(\varepsilon^k)| \\ &= \Big| \sum_{l \in I_n} c_l H_l(\varepsilon^k) - \sum_{j \in I_m} c_j H_j(\varepsilon^k) \Big| \\ &\geq (1-2\varepsilon) \sum_{l \in I_n} c_l - \sum_{j \in I_m} c_j(-1) \\ &= 2 - 2\varepsilon. \end{aligned}$$

Thus, we get that $\|\Gamma_\infty\|_{e,w} \geq 1$ and we finish the proof. □

Theorem 4.4. *Let $\gamma_\infty : \ell^\infty \longrightarrow \ell^\infty$ be the Cesàro sequence operator. We have*

$$\|\gamma_\infty\|_{e,w} = \|\gamma_\infty\|_e = \|\gamma_\infty\| = 1.$$

In particular we have $a_n(\gamma_\infty) = 1$ for any $n \in \mathbb{N}$.

Proof. The upper estimate is clear, as $\|\gamma_\infty\|_{e,w} \leq \|\gamma_\infty\|_e \leq \|\gamma_\infty\| = 1$. For the lower bound, we follow the same idea as in the proof of Theorem 4.3. We fix $\varepsilon \in (0,1)$ and we let r be a natural number with $r \geq 1/\varepsilon > 1$. For $n \in \mathbb{N}$, we consider the sequence $a^{(n)} \in \ell^\infty$ defined by $a^{(n)} = (a_k^{(n)})_{k \in \mathbb{N}^*}$, where $a_k^{(n)} = -1$ if $k \leq r^n$, and $a_k^{(n)} = 1$ if $k > r^n$. We denote by $A^{(n)} := \gamma_\infty(a^{(n)})$ with $A^{(n)} = (A_i^{(n)})_{i \in \mathbb{N}^*}$. Then, we get

$$A_i^{(n)} = \begin{cases} -1 & \text{if } i \leq r^n \\ \frac{i - 2r^n}{i} & \text{if } i > r^n. \end{cases}$$

Now, we consider a block-subsequence of $A^{(n)}$, say $(\widetilde{A}^{(m)})_m$, as in Definition 3.1 by

$$\widetilde{A}^{(m)} = \sum_{j \in I_m} c_j A^{(j)}.$$

By the choice of $(a_n)_n$, we have for two integers j, k with $j < k$, that

$$A_{r^k}^{(j)} = 1 - 2\frac{r^j}{r^k} \geq 1 - 2\varepsilon.$$

Let m, n be two integers with $m < n$, and let $k = \min I_n$. We compute

$$\begin{aligned}
\|\widetilde{A}^{(m)} - \widetilde{A}^{(n)}\|_\infty &\geq |\widetilde{A}^{(m)}_{r^k} - \widetilde{A}^{(n)}_{r^k}| \\
&= \Big| \sum_{j \in I_m} c_j A^{(j)}_{r^k} - \sum_{l \in I_n} c_l A^{(l)}_{r^k} \Big| \\
&\geq (1 - 2\varepsilon) \sum_{j \in I_m} c_j - \sum_{l \in I_n} c_l (-1) \\
&= 2 - 2\varepsilon.
\end{aligned}$$

Finally, by applying Lemma 3.4 (*ii*) we deduce $\|\gamma_\infty\|_{e,w} \geq 1$. ☐

The operators Γ_p are not compact, but the situation is different when we restrict them to a Müntz space M^p_Λ. This is due to the geometric nature of these spaces. We first state the following lemma.

Lemma 4.5. *Let* $p, q \in [1, +\infty]$ *satisfy* $p > q$. *Then the natural inclusion* $i_{p,q} : M^p_\Lambda \to M^q_\Lambda$ *is a compact operator.*

Proof. The boundedness is clear. To check the compacity, we consider a sequence $(f_n)_n$ in the unit ball of M^p_Λ. From [AL, Cor. 2.5], there exist a function $g \in M^p_\Lambda$ and an extraction $(n_k)_k$ such that f_{n_k} converges to g uniformly on every compact subset of $[0, 1)$. For any $\delta \in (0, 1)$, we compute

$$\begin{aligned}
\|f_{n_k} - g\|_q^q &= \int_0^{1-\delta} |f_{n_k}(t) - g(t)|^q dt + \int_{1-\delta}^1 |f_{n_k}(t) - g(t)|^q dt \\
&\leq \|f_{n_k} - g\|_{p,[0,1-\delta]} + \|f_{n_k} - g\|_p^{\frac{q}{p}} \delta^{1-\frac{q}{p}}.
\end{aligned}$$

In the second term, we use the Hölder inequality. Clearly, the first term tends to 0 by the uniform convergence, and the second one is less than $2^{\frac{q}{p}} \delta^{1-\frac{q}{p}}$. As $1 - \frac{q}{p} > 0$, we get that f_{n_k} converges to g in M^q_Λ, and thus $i_{p,q}$ is compact. ☐

Next, we obtain the following property for the restrictions of the Cesàro operator.

Proposition 4.6. *Let* $p \in [1, +\infty]$, M^p_Λ *be a Müntz space and* $\Gamma^\Lambda_p : M^p_\Lambda \to M^p_\Lambda$, $f \mapsto \Gamma(f)$ *be the restriction of the Cesàro operator. Then* Γ^Λ_p *is compact.*

Proof. According to [AHLM, Prop. 4.2], the operator $\Gamma_\Lambda : M^1_\Lambda \to M^\infty_\Lambda, f \mapsto \Gamma(f)$ is bounded (but not compact). Then we obtain the factorization

$$\begin{tikzcd}
M^p_\Lambda \arrow[r, "\Gamma^\Lambda_p"] \arrow[d, "i_{p,1}"'] & M^p_\Lambda \\
M^1_\Lambda \arrow[r, "\Gamma_\Lambda"'] & M^\infty_\Lambda . \arrow[u, "i_{\infty,p}"']
\end{tikzcd}$$

Therefore, Lemma 4.5 yields to the compacity of Γ^Λ_p. ☐

4.2 Cesàro operators on Cesàro spaces

In this section, we study the Cesàro operators defined on the Cesàro spaces to the corresponding Lebesgue spaces. We shall also consider the restriction of those operators to the Müntz subspaces (see Definition 2.5). We note that for $p \in [1, +\infty]$ (resp. for $p \in (1, +\infty]$) the Cesàro operators $\Gamma_{C(p)}$ (resp. $\gamma_{c(p)}$) are naturally well defined and bounded, with norm 1. Moreover, they map isometrically the set of positive functions (resp. sequences) to themselves. It is shown in [CR] that these operators are not compact and we shall show below that they are even far from being compact: their distance to compact operators is maximal.

Theorem 4.7. *For any $p \in (1, +\infty)$, we have*

(i) $\|\Gamma_{C(p)}\|_e = \|\Gamma_{C(p)}\| = 1.$

(ii) $\|\gamma_{c(p)}\|_e = \|\gamma_{c(p)}\| = 1.$

(iii) $\|\Gamma_{C(1)}\|_{e,w} = \|\Gamma_{C(1)}\|_e = \|\Gamma_{C(1)}\| = 1.$

(iv) $\|\Gamma_{C(\infty)}\|_{e,w} = \|\Gamma_{C(\infty)}\|_e = \|\Gamma_{C(\infty)}\| = 1.$

(v) $\|\gamma_{c(\infty)}\|_{e,w} = \|\gamma_{c(\infty)}\|_e = \|\gamma_{c(\infty)}\| = 1.$

In particular, the approximation numbers of all these operators, are equal to 1.

Proof. Since $\|\Gamma_{C(p)}\|_e \leq \|\Gamma_{C(p)}\| \leq 1$, all we need to check is that the essential norm is bigger than 1. Let $p \in (1, +\infty)$, then we have $\Gamma_p = \Gamma_{C(p)} \circ J_p$, where $J_p : L^p \to \mathrm{Ces}_p$ is the formal inclusion of L^p in Ces_p. It is easy to see that this factorization implies

$$\|\Gamma_p\|_e \leq \|\Gamma_{C(p)}\|_e \|J_p\|.$$

Using the Hardy inequality ([HLP, Th. 327]) and Theorem 4.1, we obtain the estimate $p' \leq p' \|\Gamma_{C(p)}\|_e$, and thus we get (i). Following the same steps, we can treat the sequential case by applying Theorem 4.2 and we obtain (ii). For the point (iii), we clearly have

$$\|\Gamma_{C(1)}\|_{e,w} \leq \|\Gamma_{C(1)}\|_e \leq \|\Gamma_{C(1)}\| \leq 1.$$

To prove the lower estimate for $\|\Gamma_{C(1)}\|_{e,w}$, we apply Proposition 3.15 for the sets $A_n = [1 - 1/n, 1]$ and for the sequence of normalized functions $(h_n) \in \mathrm{Ces}_1$ defined by $h_n(x) = (\lambda_n + 1)^2 x^{\lambda_n}$. The Lebesgue measure of the sets A_n decreases to 0 when $n \to +\infty$, and for any fixed $k \in \mathbb{N}$ we have

$$\|\Gamma_{C(1)}(h_n)\|_{L^1(A_k)} = \int_{1-\frac{1}{k}}^{1} \Big(\frac{1}{x} \int_0^x (\lambda_n + 1)^2 t^{\lambda_n} dt\Big) dx = 1 - \Big(1 - \frac{1}{k}\Big)^{\lambda_n + 1}.$$

It then tends to 1 when $n \to +\infty$, which gives $\|\Gamma_{C(1)}\|_{e,w} \geq 1$ as desired. To prove (iv), we have as usual $\|\Gamma_{C(\infty)}\|_{e,w} \leq \|\Gamma_{C(\infty)}\|_e \leq \|\Gamma_{C(\infty)}\| \leq 1$, and as in the proof of (i), Theorem 4.3 gives

$$\|\Gamma_{C(\infty)}\|_{e,w} \|J_\infty\| \geq \|\Gamma_{C(\infty)} \circ J_\infty\|_{e,w} = \|\Gamma_\infty\|_{e,w} = 1.$$

In the same way, we treat the sequential case (v) with Theorem 4.4. □

Now we consider the restrictions of Cesàro-type operators to Müntz Cesàro spaces (see Definition 2.5). Let $\Lambda = (\lambda_n)_{n\geq 0}$ be an increasing sequence satisfying the Müntz and gap-conditions. For $p \in [1, +\infty]$, we define the following operator

$$\Gamma^{\Lambda}_{C(p)} : \left\{ \begin{array}{ccc} M_{\Lambda}^{\mathrm{Ces}_p} & \longrightarrow & M^p_{\Lambda} \\ f & \longmapsto & \Gamma(f). \end{array} \right.$$

Theorem 4.8. *Let $p \in [1, +\infty)$. Then we have*

(*i*) $\|\Gamma^{\Lambda}_{C(p)}\|_e = 1$.

(*ii*) $\|\Gamma^{\Lambda}_{C(\infty)}\|_e = \dfrac{1}{2}$.

Proof. First, we prove (*i*). The operator $\Gamma^{\Lambda}_{C(p)}$ is clearly well defined and bounded. We have $\|\Gamma^{\Lambda}_{C(p)}\|_e \leq \|\Gamma^{\Lambda}_{C(p)}\| \leq \|\Gamma_{C(p)}\| \leq 1$. Let us denote $\chi_p : M_{\Lambda}^{\mathrm{Ces}_p} \to L^p$, $f \mapsto \Gamma(f)$. We factorize χ_p through M^p_{Λ} as follows

$$\chi_p = j_p \circ \Gamma^{\Lambda}_{C(p)},$$

where $j_p : M^p_{\Lambda} \to L^p$ is the inclusion of M^p_{Λ} in L^p. Hence, we obtain

$$\|\chi_p\|_e \leq \|\Gamma^{\Lambda}_{C(p)}\|_e . \|j_p\| \leq \|\Gamma^{\Lambda}_{C(p)}\|_e,$$

and we just need to check that $\|\chi_p\|_e \geq 1$. The operator χ_p is valued in an L^p space and therefore we can apply Corollary 3.9 with $\Omega = [0, 1]$, $A_k = [1-1/k, 1]$, $\alpha = 1$ and with the sequence of functions $h_n : t \mapsto (\lambda_n + 1)(p\lambda_n + 1)^{\frac{1}{p}} t^{\lambda_n}$. We have for any fixed $k \in \mathbb{N}$,

$$\begin{aligned} \|\chi_p(h_n)\|^p_{L^p(A_k)} &= \int_{1-\frac{1}{k}}^{1} (p\lambda_n + 1)\Big(\frac{1}{x}\int_0^x (\lambda_n + 1)t^{\lambda_n}dt\Big)^p dx \\ &= 1 - \Big(1 - \frac{1}{k}\Big)^{p\lambda_n+1}. \end{aligned}$$

We obtain $\|\Gamma^{\Lambda}_{C(p)}\|_e \geq \|\chi_p\|_e \geq 1$ and the proof of (*i*) is complete. Now we treat the case "$p = +\infty$". For the upper estimate of the essential norm, we factorize $\Gamma^{\Lambda}_{C(\infty)}$ through M^1_{Λ} as follows

$$\begin{tikzcd} M_{\Lambda}^{\mathrm{Ces}_\infty} \arrow[rr, "\Gamma^{\Lambda}_{C(\infty)}"] \arrow[dr, "J_\Lambda"'] & & M^\infty_{\Lambda} \\ & M^1_{\Lambda} \arrow[ur, "\Gamma_\Lambda"'] & \end{tikzcd}$$

where $J_\Lambda : M_{\Lambda}^{\mathrm{Ces}_\infty} \to M^1_{\Lambda}$ is the restriction of the inclusion of Ces_∞ in L^1 (see Lemma 2.1) and $\Gamma_\Lambda : M^1_{\Lambda} \to M^\infty_{\Lambda}$ is the Cesàro operator between Müntz spaces. By [AHLM, Thm. 4.3], we have that $\|\Gamma_\Lambda\|_e = 1/2$, and thus we obtain

$$\|\Gamma^{\Lambda}_{C(\infty)}\|_e \leq \|J_\Lambda\| . \|\Gamma_\Lambda\|_e \leq \frac{1}{2}.$$

To get the lower estimate, we consider a subsequence $(\gamma_n)_n \subset \Lambda$ satisfying $\frac{\gamma_{n+1}}{\gamma_n} \to +\infty$ and denote $(f_n)_n \in M_\Lambda^{\mathrm{Ces}_\infty}$ the sequence of normalized functions defined by $f_n(x) = (\gamma_n + 1)x^{\gamma_n}$. For $m > n$, we have

$$\begin{aligned}\|\Gamma(f_n) - \Gamma(f_m)\|_\infty &= \|x^{\gamma_n} - x^{\gamma_m}\|_\infty \\ &= \Big(\frac{\gamma_n}{\gamma_m}\Big)^{\gamma_n/(\gamma_m - \gamma_n)} - \Big(\frac{\gamma_n}{\gamma_m}\Big)^{\gamma_m/(\gamma_m - \gamma_n)} \\ &= \Big(\frac{\gamma_n}{\gamma_m}\Big)^{\gamma_n/(\gamma_m - \gamma_n)} \Big(1 - \frac{\gamma_n}{\gamma_m}\Big).\end{aligned}$$

As this term tends to 1 when $n, m \to +\infty$ with $n < m$, we get *(ii)* from Lemma 3.4. ☐

The previous result implies that the operators $\Gamma_{C(p)}^\Lambda$ are never compact. Now we focus on the particular case where Λ is lacunary to obtain more specific results. Recall that a sequence $(\lambda_n)_n$ is called *lacunary* if it satisfies $\inf_{n\geq 0} \frac{\lambda_{n+1}}{\lambda_n} > 1$.

Theorem 4.9. *Let $p \in [1, +\infty)$. If $\Lambda = (\lambda_n)_n$ is a lacunary sequence, then the operator $\Gamma_{C(p)}^\Lambda : M_\Lambda^{\mathrm{Ces}_p} \to M_\Lambda^p$ is an isomorphism. Actually, there exist two constant $C_1, C_2 \in \mathbb{R}_+^*$ such that for any $b = (b_n)_n \in c_0$ we have*

$$C_1\Big(\sum_n \frac{|b_n|^p}{\lambda_n^{1+p}}\Big)^{\frac{1}{p}} \leq \Big\|\sum_n b_n t^{\lambda_n}\Big\|_{C(p)} \leq C_2\Big(\sum_n \frac{|b_n|^p}{\lambda_n^{1+p}}\Big)^{\frac{1}{p}}.$$

Proof. Using the Gurariy-Macaev theorem in L^p (see [GL, Th. 9.3.3]), there exist two positive numbers d_1 and d_2 such that

$$d_1\Big(\sum_{n=0}^{\infty} \frac{|a_n|^p}{\lambda_n}\Big)^{\frac{1}{p}} \leq \|g\|_p \leq d_2\Big(\sum_{n=0}^{\infty} \frac{|a_n|^p}{\lambda_n}\Big)^{\frac{1}{p}},$$

for any function $g \in M(\Lambda)$ with the form $g(t) = \sum_n a_n t^{\lambda_n}$. Therefore, for any function $f \in M(\Lambda)$ defined by $f(t) = \sum_n b_n t^{\lambda_n}$, we get from the one hand

$$\begin{aligned}\|\Gamma(f)\|_p^p \leq \|f\|_{C(p)}^p &\leq \int_0^1 \Big(\frac{1}{x}\int_0^x \sum_n |b_n| t^{\lambda_n} dt\Big)^p dx \\ &= \Big\|\sum_n \frac{|b_n|}{\lambda_n + 1} x^{\lambda_n}\Big\|_p^p \\ &\leq d_2^p \sum_n \frac{|b_n|^p}{\lambda_n^{1+p}},\end{aligned}$$

since $\sum_n |b_n| t^{\lambda_n} \in M(\Lambda)$. On the other hand, as $\Gamma(f) \in M(\Lambda)$ we write

$$\begin{aligned}\|\Gamma(f)\|_p^p &= \Big\|\sum_n \frac{b_n}{\lambda_n + 1} x^{\lambda_n}\Big\|_p^p \\ &\geq \Big(\frac{d_1}{2}\Big)^p \sum_n \frac{|b_n|^p}{\lambda_n^{1+p}}.\end{aligned}$$

Hence, we find the claimed estimates. In particular, the operator $\Gamma_{C(p)}^\Lambda$ is one-to-one, with a closed range. Since $\Gamma_{C(p)}^\Lambda(M(\Lambda)) = M(\Lambda)$, it has a dense range in M_Λ^p and therefore $\Gamma_{C(p)}^\Lambda$ is an isomorphism. ☐

Remark 4.10. Note that the Gurariy-Macaev theorem in L^1 and Remark 2.6 imply that $M_\Lambda^{\mathrm{Ces}_\infty}$ is isomorphic to ℓ^1 (in the case where Λ is lacunary). Hence, we get

$$\Big\| \sum_n b_n t^{\lambda_n} \Big\|_{C(\infty)} \approx \sum_n \frac{|b_n|}{\lambda_n},$$

where the underlying constants depend only on Λ. Moreover, even in the lacunary case, Γ cannot be an isomorphism between $M_\Lambda^{\mathrm{Ces}_\infty}$ and M_Λ^∞ since the spaces are not isomorphic. A natural question that arises in this context: Is Γ an isomorphism between $M_\Lambda^{\mathrm{Ces}_p}$ and M_Λ^p for any Müntz sequence Λ and any $p \in [1, +\infty)$?

5 Multiplication operators on Cesàro function spaces

In this section, we study the compactness and compute the essential norm of the multiplication operator $T_\psi : f \mapsto f\psi$ on the Cesàro function spaces, for a measurable bounded function ψ on $[0, 1]$. The starting point in this part is the following result:

Proposition 5.1. *[AMR, Theorem 2.1] Let $p \in [1, +\infty]$ and assume that ψ is a measurable function on $[0, 1]$. Then the following are equivalent:*

- *(i) The multiplication operator $T_\psi : \mathrm{Ces}_p \to \mathrm{Ces}_p$, $f \mapsto f\psi$ is well defined.*
- *(ii) The operator T_ψ is bounded.*
- *(iii) The function ψ is essentially bounded on $[0, 1]$.*

Moreover, in this case we have $\|T_\psi\| = \|\psi\|_\infty$.

Formally, this result was proved when p is finite. Nevertheless, the proof can be easily adapted for $p = +\infty$. We can use the method with our framework to compute the essential norm of the multiplication operators.

Theorem 5.2. *Let $\psi \in L^\infty([0, 1])$ and $T_\psi : \mathrm{Ces}_p([0, 1]) \to \mathrm{Ces}_p([0, 1])$; $f \mapsto f\psi$ be the multiplication operator. Then we have*

$$\|T_\psi\|_e = \|\psi\|_\infty.$$

Proof. As usual, we have $\|T_\psi\|_e \le \|T_\psi\| = \|\psi\|_\infty$ by Proposition 5.1, and hence we just need to check that $\|T_\psi\|_e \ge \|\psi\|_\infty$. For $\varepsilon > 0$, we define the set

$$A_\varepsilon = \{t \in [0, 1], |\psi(t)| \ge \|\psi\|_\infty - \varepsilon\}.$$

Let μ be the Lebesgue measure. As $\mu(A_\varepsilon) > 0$, then at least one of the two sets $[0, \frac{1}{2}] \cap A_\varepsilon$ or $[\frac{1}{2}, 1] \cap A_\varepsilon$ has a striclty positive measure. Assume that it is the first one, and put

$$\beta = \inf\{x \in \big[0, 1/2\big], \mu(\big[x, 1/2\big] \cap A_\varepsilon) = 0\}.$$

The number β satisfies $\beta \in (0, 1)$. In the other case, we define

$$\beta' = \sup\{x \in [1/2, 1], \mu([1/2, x] \cap A_\varepsilon) = 0\},$$

and β' is also in $(0,1)$. Now we consider an increasing sequence (a_n) which tends to β when $n \to +\infty$, and define the sets $J_n = [a_n, a_{n+1}) \cap A_\varepsilon$ for any integer $n \in \mathbb{N}$. From the definition of β, there exist infinitely many sets J_n with a positive Lebesgue measure. Up to an extraction, we can assume that they all satisfy $\mu(J_n) > 0$. We then set the normalized functions $f_n = \dfrac{\mathbb{1}_{J_n}}{\|\mathbb{1}_{J_n}\|_{C(p)}} \in \mathrm{Ces}_p$. Assume first that p is finite. For $n < m$, we have

$$\begin{aligned}\|T_\psi(f_n) - T_\psi(f_m)\|_{C(p)}^p &= \int_{a_n}^1 \Big(\frac{1}{x}\int_0^x |\psi(t)|.\Big|\frac{\mathbb{1}_{J_n}(t)}{\|\mathbb{1}_{J_n}\|} - \frac{\mathbb{1}_{J_m}(t)}{\|\mathbb{1}_{J_m}\|}\Big|dt\Big)^p dx \\ &\geq (\|\psi\|_\infty - \varepsilon)^p \int_\beta^1 \Big(\frac{1}{x}\int_0^x \Big(\frac{\mathbb{1}_{J_n}(t)}{\|\mathbb{1}_{J_n}\|} + \frac{\mathbb{1}_{J_m}(t)}{\|\mathbb{1}_{J_m}\|}\Big)dt\Big)^p dx \\ &= (\|\psi\|_\infty - \varepsilon)^p \Big(\frac{\mu(J_n)}{\|\mathbb{1}_{J_m}\|} + \frac{\mu(J_m)}{\|\mathbb{1}_{J_m}\|}\Big)^p \int_\beta^1 \frac{dx}{x^p}.\end{aligned}$$

The lower bound follows from the fact $J_n \subset A_\varepsilon$ and from the disjointness of the intervals J_n, and the equality holds because $\beta > \sup J_n$ for any n. On the other hand, we have

$$\|\mathbb{1}_{J_n}\|_{C(p)}^p = \int_{a_n}^\beta \Big(\frac{1}{x}\mu([0,x] \cap J_n)\Big)^p dx + \mu(J_n)^p \int_\beta^1 \frac{dx}{x^p},$$

and since $a_n \to \beta < 1$, we easily obtain

$$\lim_{n\to+\infty} \frac{\mu(J_n)}{\|\mathbb{1}_{J_n}\|_{C(p)}} = \Big(\int_\beta^1 \frac{dx}{x^p}\Big)^{-\frac{1}{p}}.$$

Hence there exists $n_0 \in \mathbb{N}$ such that for any $m > n \geq n_0$ we have

$$\|T_\psi(f_n) - T_\psi(f_m)\|_{C(p)} \geq (2-\varepsilon)\big(\|\psi\|_\infty - \varepsilon\big).$$

This holds for any $\varepsilon > 0$ and Lemma 3.4 gives $\|T_\psi\|_e \geq \|\psi\|_\infty$. This concludes the proof when p is finite. Assume now that $p = +\infty$ and fix again two integers n, m with $n < m$,

$$\begin{aligned}\|T_\psi(f_n) - T_\psi(f_m)\|_{C(\infty)} &= \sup_{x\in(0,1]} \frac{1}{x}\int_0^x |\psi(t)|.\Big|\frac{\mathbb{1}_{J_n}(t)}{\|\mathbb{1}_{J_n}\|} - \frac{\mathbb{1}_{J_m}(t)}{\|\mathbb{1}_{J_m}\|}\Big|dt \\ &\geq \big(\|\psi\|_\infty - \varepsilon\big) \sup_{x\in(0,1)} \frac{1}{x}\int_0^x \Big(\frac{\mathbb{1}_{J_n}(t)}{\|\mathbb{1}_{J_n}\|} + \frac{\mathbb{1}_{J_m}(t)}{\|\mathbb{1}_{J_m}\|}\Big)dt.\end{aligned}$$

Here, we use the fact that the intervals J_n are disjoints and included in A_ε. For any $k \in \mathbb{N}$, the set J_k satisfies $\inf(J_k) = a_k$, and this gives

$$\|\mathbb{1}_{J_k}\|_{C(\infty)} \leq \frac{1}{a_k}\mu(J_k).$$

Thus, we find

$$\begin{aligned}\|T_\psi(f_n) - T_\psi(f_m)\|_{C(\infty)} &\geq (\|\psi\|_\infty - \varepsilon)\frac{1}{a_{m+1}}\int_0^{a_{m+1}} \Big(\frac{\mathbb{1}_{J_n}(t)}{\|\mathbb{1}_{J_n}\|} + \frac{\mathbb{1}_{J_m}(t)}{\|\mathbb{1}_{J_m}\|}\Big)dt \\ &= (\|\psi\|_\infty - \varepsilon)\frac{1}{a_{m+1}}\Big(\frac{\mu(J_n)}{\|\mathbb{1}_{J_n}\|} + \frac{\mu(J_m)}{\|\mathbb{1}_{J_m}\|}\Big) \\ &\geq (\|\psi\|_\infty - \varepsilon)\frac{a_m + a_n}{a_{m+1}}.\end{aligned}$$

As $m, n \to +\infty$ with $n < m$, we have $a_n, a_m, a_{m+1} \to \beta > 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that the sequence $(T(f_n))_{n \geq n_0}$ is $(2-\varepsilon)(\|\psi\|_\infty - \varepsilon)$-separated, and we deduce the lower estimate by applying Lemma 3.4. □

Now we are interested in the restriction of the multiplication operators to the Müntz subspaces of Ces_∞.

Lemma 5.3. *Let* $p \in [1, +\infty]$, $\Lambda = (\lambda_n)_n$ *be a sequence satisfying the Müntz and gap conditions,* $\psi \in L^\infty$ *be a function such that* $\lim\limits_{a \to 1} \|\psi \mathbb{1}_{[a,1]}\|_\infty = 0$. *Then the restriction of the multiplication operator to the Müntz-Cesàro space* $T_{\psi,\Lambda} : M_\Lambda^{\mathrm{Ces}_p} \to \mathrm{Ces}_p$ *defined by* $T_\psi(f) = f\psi$ *is compact.*

Proof. Let $(f_n)_n$ be a sequence in the unit ball of $M_\Lambda^{\mathrm{Ces}_p}$ and $\varepsilon > 0$. Since ψ is continuous and $\psi(1) = 0$, there exists $\delta \in (0, \frac{1}{2})$ such that $|\psi(t)| \leq \varepsilon$ for almost every $t \in [1-\delta, 1]$. By Corollary 2.9, there exist a function f in the unit ball of Ces_p and a subsequence $(f_{n_k})_k$ that converges uniformly to f on $[0, 1-\delta]$. Assume fisrst that $p = +\infty$, then we have

$$
\begin{aligned}
\|T_\psi(f_{n_k}) - T_\psi(f)\|_{C(\infty)} &= \sup_{x \in (0,1]} \Big(\frac{1}{x} \int_0^x |f_{n_k}(t) - f(t)|.|\psi(t)| dt \Big) \\
&= \max \Bigg\{ \sup_{x \in (0,1-\delta]} \Big(\frac{1}{x} \int_0^x |f_{n_k}(t) - f(t)|.|\psi(t)| dt \Big) \;, \\
&\qquad \sup_{x \in [1-\delta,1]} \Big(\frac{1}{x} \int_0^x |f_{n_k}(t) - f(t)|.|\psi(t)| dt \Big) \quad \Bigg\} \\
&\leq \|\psi\|_\infty \big\| f_{n_k} - f \big\|_{[0,1-\delta]} + \|\psi\|_{[1-\delta,1]} \sup_{x \in [1-\delta,1]} \Big(\frac{1}{x} \int_0^x |f_{n_k}(t) - f(t)| dt \Big) \\
&\leq \|\psi\|_\infty \big\| f_{n_k} - f \big\|_{[0,1-\delta]} + \varepsilon \big\| f_{n_k} - f \big\|_{C(\infty)}
\end{aligned}
$$

Since (f_{n_k}) converges uniformly to f on the compact set $[0, 1-\delta]$ and both f_{n_k} and f have norm less than 1, we obtain

$$
\lim_{k \to \infty} \|T_\psi(f_{n_k}) - T_\psi(f)\|_{C(\infty)} \leq 2\varepsilon,
$$

and so $T_{\psi,\Lambda}$ is a compact operator on $M_\Lambda^{\mathrm{C}(\infty)}$. Assume now that p is finite, we have

$$
\begin{aligned}
\|T_\psi(f_{n_k}) - T_\psi(f)\|_{C(p)}^p &= \int_0^1 \Big(\frac{1}{x} \int_0^x |f_{n_k}(t) - f(t)|.|\psi(t)| dt \Big)^p dx \\
&\leq \int_0^{1-\delta} \Big(\frac{1}{x} \int_0^x |f_{n_k} - f|.|\psi(t)| dt \Big)^p dx \\
&\quad + \int_{1-\delta}^1 \left(\frac{1}{1-\delta} \int_0^{1-\delta} |f_{n_k} - f|.|\psi(t)| dt + \varepsilon \frac{1}{x} \int_{1-\delta}^x |f_{n_k} - f| dt \right)^p dx,
\end{aligned}
$$

as $|\psi(t)| \leq \varepsilon$ when $t \geq 1-\delta$. On the other hand, for any $x \leq 1-\delta$, we have

$$
\frac{1}{x} \int_0^x \big| f_{n_k}(t) - f(t) \big|.|\psi(t)| dt \leq \|\psi\|_\infty \big\| f_{n_k} - f \big\|_{[0,1-\delta]}.
$$

Using the estimate $(A+B)^p \leq 2^p(A^p+B^p)$ for $A, B \geq 0$, we find

$$\begin{aligned}\|T_\psi(f_{n_k}) - T_\psi(f)\|_{C(p)}^p &\leq \|\psi\|_\infty^p \big\|f_{n_k} - f\big\|_{[0,1-\delta]}^p \Big(1 + \frac{2^p\delta}{(1-\delta)^p}\Big) \\ &\quad + 2^p\varepsilon^p \int_{1-\delta}^1 \Big(\frac{1}{x}\int_{1-\delta}^x |f_{n_k} - f|dt\Big)^p dx \\ &\leq C\|\psi\|_\infty . \big\|f_{n_k} - f\big\|_{[0,1-\delta]} + 2^{p+1}\varepsilon^p.\end{aligned}$$

Since $\|f_{n_k} - f\|_{[0,1-\delta]} \to 0$ when $k \to +\infty$, we deduce that $T_\psi(f_{n_k}) \to T_\psi(f)$ in Ces_p when $k \to +\infty$ and hence $T_{\psi,\Lambda}$ is compact on $M_\Lambda^{C(p)}$. ☐

Note that the assumption on ψ in Theorem 5.3 is satisfied if ψ is continuous at the point 1 and satisfies $\psi(1) = 0$.

Theorem 5.4. *Let Λ be an increasing sequence satisfying the Müntz and gap-conditions, $p \in [1, +\infty]$, $\psi \in L^\infty$ and $T_{\psi,\Lambda} : M_\Lambda^{C(p)} \to \mathrm{Ces}_p$ be the multiplication operator defined in Lemma 5.3. If ψ is continuous at the point 1, then we have*

$$\|T_{\psi,\Lambda}\|_e = |\psi(1)|.$$

Proof. For any $n \in \mathbb{N}$, we let $\psi_n = \psi(t)f_n(t)$ where

$$f_n(t) = \begin{cases} 1 & \text{if } t \in [0, 1-\frac{1}{n}] \\ n(1-t) & \text{if } t \in [1-\frac{1}{n}, 1] \end{cases}$$

is a continuous function with $f_n(1) = 0$. Since $\psi_n(1) = 0$, we know from Lemma 5.3 that $T_{\psi_n,\Lambda}$ is compact for any n. Hence,

$$\|T_{\psi,\Lambda}\|_e \leq \|T_{\psi,\Lambda} - T_{\psi_n,\Lambda}\| \leq \|\psi - \psi_n\|_\infty \leq \|\psi\|_{[1-\frac{1}{n},1]} \underset{n\to\infty}{\to} |\psi(1)|,$$

as ψ is continuous at 1. To get the lower estimate, we will apply Lemma 3.4. For this, we let $\varepsilon > 0$. Since ψ is continuous at 1, there exists $\delta \in (0,1)$ such that for any $t \in [1-\delta, 1]$, we have $|\psi(t)| \geq (1-\varepsilon)|\psi(1)|$. Assume first that $p = +\infty$ and consider a subsequence $(\gamma_n)_n \subset \Lambda$ which satisfies $\lim\limits_{n\to+\infty} \frac{\gamma_{n+1}}{\gamma_n} = +\infty$. We define the norm-one functions $(\varphi_n)_n \in M_\Lambda^{\mathrm{Ces}_\infty}$ by $\varphi_n(x) = (\gamma_n+1)x^{\gamma_n}$. Applying [A2, Lemma 3.1] for the polynomials $p(x) = q(x) = x$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\varphi_n\|_1 + \|\varphi_m\|_1 \leq (1+\varepsilon)\int_{1-\delta}^1 |\varphi_n(t) - \varphi_m(t)|dt,$$

for any $m > n \geq n_0$. One can also check this estimate by a straightforward computation of $\|\varphi_n - \varphi_m\|_1$, using the assumption $\frac{\gamma_{n+1}}{\gamma_n} \to +\infty$ when $n \to +\infty$. We get

$$\begin{aligned}\|T_\psi(\varphi_n) - T_\psi(\varphi_m)\|_{C(\infty)} &= \|\Gamma\left(|\psi(\varphi_n - \varphi_m)|\right)\|_\infty \\ &= \sup_{x\in(0,1]} \frac{1}{x}\int_0^x |(\varphi_n(t) - \varphi_m(t))\psi(t)|dt \\ &\geq \int_{1-\delta}^1 |(\varphi_n(t) - \varphi_m(t))\psi(t)|dt \\ &\geq 2|\psi(1)|\frac{1-\varepsilon}{1+\varepsilon}.\end{aligned}$$

By Lemma 3.4, we find $\|T_{\psi,\Lambda}\|_e \geq |\psi(1)|\frac{1-\varepsilon}{1+\varepsilon}$ for any $\varepsilon > 0$, and therefore we deduce $\|T_{\psi,\Lambda}\| = |\psi(1)|$ in this case. Assume now that p is finite. We will follow the method of the proof of [GL2]. Let $\gamma = (\gamma_n)_{n\in\mathbb{N}}$ be a subsequence of Λ which satisfies

$$\forall n \in \mathbb{N}, \quad \gamma_{n+1} + 1 \geq \big(\gamma_n + 1\big)^6.$$

We set a sequence of disjoints intervals $J_k = (\alpha_k, \beta_k)$ with

$$\alpha_k = \exp\Big(-\frac{1}{(\gamma_k+1)^{\frac{1}{2}}}\Big) \quad \text{and} \quad \beta_k = \exp\Big(-\frac{1}{(\gamma_k+1)^3}\Big).$$

The numbers α_k, β_k satisfy for any $k \in \mathbb{N}$, $\alpha_k \leq \beta_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \cdots$. For $k \in \mathbb{N}$, we define $\varphi_k(t) = (\gamma_n + 1)(p\gamma_n + 1)^{1/p} t^{\gamma_k}$. The sequence $(\varphi_k)_k \in M_\Lambda^{C(p)}$ is normalized, and each function φ_k is concentrated on the interval J_k in the following sense: if $a < b \leq \alpha_k$, we have

$$\begin{aligned}
\int_a^b \varphi_k(t)dt &= \big(b^{\gamma_k+1} - a^{\gamma_k+1}\big)(p\gamma_k+1)^{\frac{1}{p}} \\
&\leq \alpha_k^{\gamma_k+1}(p\gamma_k+1)^{\frac{1}{p}} \\
&\leq \exp\big(-(\lambda_k+1)^{\frac{1}{2}}\big)(p\lambda_k+1)^{\frac{1}{p}} \to 0,
\end{aligned}$$

when $k \to +\infty$. On the other hand, if $\beta_k \leq c < d$, we write

$$\begin{aligned}
\int_c^d \varphi_k(t)dt &= \big(d^{\gamma_k+1} - c^{\gamma_k+1}\big)(p\gamma_k+1)^{\frac{1}{p}} \\
&\leq \big(1 - \beta_k^{\gamma_k+1}\big)(p\gamma_k+1)^{\frac{1}{p}} \\
&\leq \frac{(p\gamma_k+1)^{\frac{1}{p}}}{(\gamma_k+1)^2} \to 0,
\end{aligned}$$

when $k \to +\infty$. Here, we use the estimate $1 - \exp(-u) \leq u$ when $u \in (0,1)$. We also have

$$\varphi_k(\alpha_k) = (p\gamma_k+1)^{\frac{1}{p}}(\gamma_k+1)\exp\Big(-\frac{\gamma_k}{(\gamma_k+1)^{\frac{1}{2}}}\Big) \to 0,$$

when $k \to +\infty$, and

$$(1-\beta_k)\varphi_k(1) \leq \frac{(p\gamma_k+1)^{\frac{1}{p}}(\gamma_k+1)}{(\gamma_k+1)^3} \to 0,$$

when $k \to +\infty$. We define ε_k to be the maximum between these four quantities. Clearly $(\varepsilon_k)_k$ tends to 0 when $k \to +\infty$. Therefore, for any $k \in \mathbb{N}$, we compute

$$\begin{aligned}
1 = \|\varphi_k\|_{C(p)}^p = \int_0^{\alpha_k} \big(\Gamma(\varphi_k)(x)\big)^p dx &+ \int_{\alpha_k}^{\beta_k} \frac{1}{x^p}\Big(\int_0^{\alpha_k} \varphi_k(t)dt + \int_{\alpha_k}^x \varphi_k(t)dt\Big)^p dx \\
&+ \int_{\beta_k}^1 \big(\Gamma(\varphi_k)(x)\big)^p dx \\
&\leq \varepsilon_k^p + \int_{\alpha_k}^{\beta_k} \frac{1}{x^p}\Big(\varepsilon_k + \int_{\alpha_k}^x \varphi_k(t)dt\Big)^p dx + \varepsilon_k^p \\
&\sim \int_{\alpha_k}^{\beta_k} \frac{1}{x^p}\Big(\int_{\alpha_k}^x \varphi_k(t)dt\Big)^p dx,
\end{aligned}$$

when $k \to +\infty$. Now we fix n_0 such that $\alpha_{n_0} \geq 1-\delta$. Then for any $m > n \geq n_0$ we have

$$\begin{aligned}
&\|T_\psi(\varphi_n) - T_\psi(\varphi_m)\|^p_{C(p)} = \int_0^1 \Gamma(|\psi|.|\varphi_n - \varphi_m|)(x)^p dx \\
&\geq \int_{\alpha_n}^{\beta_n} \Big(\frac{1}{x}\int_{\alpha_n}^{x} |\psi|.|\varphi_n - \varphi_m|\Big)^p dx + \int_{\alpha_m}^{\beta_m} \Big(\frac{1}{x}\int_{\alpha_m}^{x} |\psi|.|\varphi_n - \varphi_m|\Big)^p dx \\
&\geq |\psi(1)|(1-\varepsilon)\Bigg(\int_{\alpha_n}^{\beta_n} \frac{1}{x^p}\Big(\int_{\alpha_n}^{x} \varphi_n(t)dt - \varepsilon_m\Big)^p dx \\
&\quad + \int_{\alpha_m}^{\beta_m} \frac{1}{x^p}\Big(\int_{\alpha_m}^{x} \varphi_m(t)dt - \varepsilon_n\Big)^p dx\Bigg) \\
&\sim 2|\psi(1)|(1-\varepsilon),
\end{aligned}$$

when $n, m \to +\infty$ with $n < m$, and we deduce the lower estimate for $\|T_{\psi,\Lambda}\|_e$. □

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